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PROPIEDADES DE ESTRELLAS RELATIVISTAS EN ROTACIÓN

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Bajo la dirección del doctor

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Memoria de Tesis Doctoral
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para optar al grado de
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Resumen

En la primera parte de esta tesis, se derivan propiedades fundamentales de modelos estelares relativistas en equilibrio (estacionarios axisimétricos asintóticamente planos y libres de convección) con rotación diferencial; en concreto, se demuestra que para una gran clase de leyes de rotación (compatible con las ecuaciones de campo y físicamente relevante) la distribución de velocidad angular del fluido tiene signo, y además la velocidad de arrastre rotacional y la densidad de momento angular tienen el mismo signo.

En el límite de rotación lenta, donde las ecuaciones de campo todavía no restringen el perfil de rotación (a través de una ley de rotación dada), se derivan condiciones suficientes que garantizan la positividad de la densidad de momento angular.

Además, el "valor medio" (con respecto a una densidad intrínseca) de la velocidad de arrastre se demuestra menor que el valor medio de la velocidad angular del fluido (independientemente de la ley de rotación, completamente en general); esta desigualdad conduce a la positividad y una cota superior de la energía total de rotación.

En la segunda parte, se estudian varias propiedades (geométricas cinemáticas y dinámicas) de dos soluciones exactas interiores, dadas por Wahlquist y por Kramer, de las ecuaciones de campo de Einstein representando el campo gravitatorio interior debido a un cuerpo de fluido perfecto axisimétrico y autogravitante en rotación estacionaria y rígida.

A pesar de las características aparentemente no-newtonianas de la superficie borde del fluido de la solución de Kramer, se demuestra, mediante un análisis detallado de las geodésicas 3-dimensionales espaciales (interiores), que sí se dan las propiedades newtonianas de convexidad.

Diferentes procedimientos ilustran los efectos 'anti-intuitivos' (desde un punto de vista newtoniano) de la dinámica del movimiento circular en estas soluciones. Las propiedades dinámicas sobre la superficie borde del fluido y la elipticidad "interior" de esta superficie son analizadas variando la velocidad de rotación de la fuente.

Abstract

In the first part of this thesis, some fundamental properties of general relativistic equilibrium stellar models (stationary axisymmetric asymptotically flat and convection-free) with differential rotation are derived; namely, it is shown that for a wide class of rotation laws (compatible with the field equations and physically relevant) the distribution of angular velocity of the fluid has a sign, and also the rate of rotational dragging and the angular momentum density have the same sign.

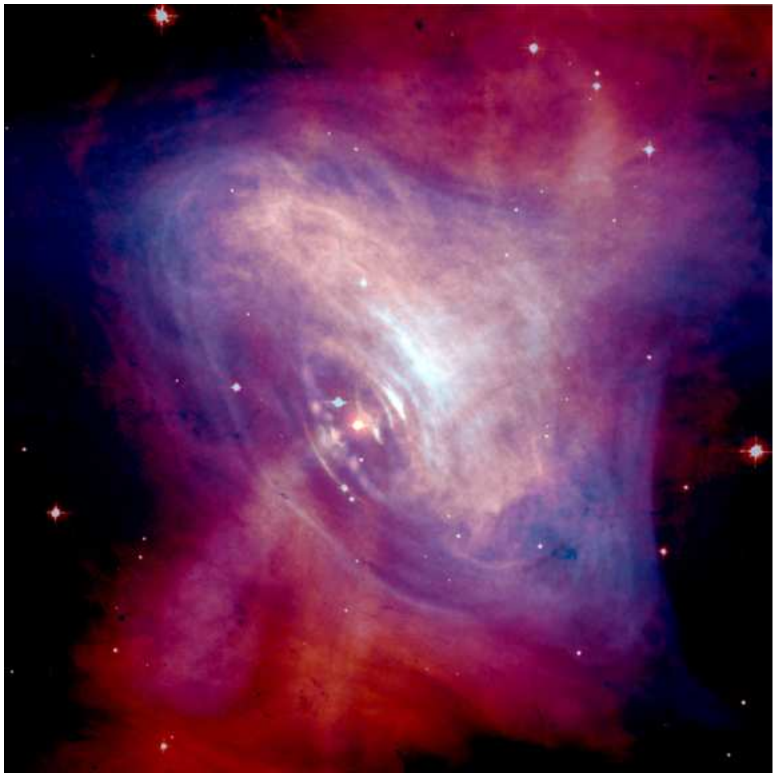
In the limit of slow rotation, where the field equations do not yet restrict the rotation profile (through a given rotation law), sufficient conditions which guarantee the positivity of the angular momentum density are deduced.

In addition, the "mean value" (with respect to an intrinsic density) of the dragging rate is shown to be less than the mean value of the fluid angular velocity (in full general, without having to restrict the rotation law); this inequality yields the positivity and an upper bound of the total rotational energy.

In the second part, several properties (geometric kinematic and dynamic) of two exact interior solutions, given by Wahlquist and by Kramer, of Einstein's field equations representing the interior gravitational field due to a self-gravitating axisymmetric body of perfect fluid in stationary and rigid rotation are studied.

In spite of the seemingly non-Newtonian features of the bounding surface of the Kramer solution for some rotation rates, it is shown, by means of a detailed analysis of the three-dimensional spatial (interior) geodesics, that the standard Newtonian convexity properties do hold.

Several procedures illustrate the 'counter-intuitive' (from a Newtonian point of view) effects of the dynamics of circular motion in these solutions. Dynamic features on the bounding surface together with the "interior" ellipticity of this surface are analyzed varying the rotation rate of the source.



Púlsar de la Nebulosa del Cangrejo. En la región central de la nebulosa (remanente de una explosión supernova, observada por astrónomos chinos en el año 1054) existe un púlsar, una estrella de neutrones (que permanece de la estrella original) girando sobre su eje 30 veces por segundo. La imagen, que combina datos ópticos de Hubble (en rojo) e imágenes de rayos X de Chandra (en azul), muestra gases nebulares alrededor del púlsar removidos por su campo magnético, partículas de alta energía y radiación emitidas por el púlsar.

"Der Wahrheit ist allezeit nur ein kurzes Siegesfest beschieden zwischen den beiden langen Zeiträumen, wo sie als paradox und als trivial gering geschätzt wird."

("A la Verdad se le concede siempre sólo un instante de triunfo, entre los dos grandes intervalos de tiempo en que se la degrada a paradoja y a trivialidad.")

Arthur Schopenhauer

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Índice general

Capítulo 1

Introducción

El estudio de estrellas en rotación en el marco de la teoría de la relatividad general es de gran interés astrofísico por varias razones, quizás la más importante es la existencia de púlsares, que son estrellas de neutrones en rotación.

La *rotación diferencial* es un estado intermedio en la evolución de una estrella hacia un estado final de equilibrio termodinámico en el que la rotación tiende a hacerse *rígida*, uniforme en todo el fluido estelar; es decir, la distribución de velocidad angular (el perfil de rotación) del fluido, debido a la fricción de unas partículas del fluido con otras, tiende a hacerse plana en el estado final de la vida de la estrella. Según se conoce actualmente, una estrella, después de haber quemado todo su combustible nuclear, se aproxima a su configuración final de equilibrio bien como una estrella "común" que contiene materia degenerada —esto es, una enana blanca, una estrella de neutrones, o incluso una "estrella extraña" conteniendo materia quark—, o bien, si tiene demasiada masa, puede acabar violentamente formando un agujero negro; sobre estas configuraciones finales singulares de agujero negro, en la actualidad muy conocidas, no hablaremos en el presente trabajo.

La rotación diferencial del Sol, estrella de mediana edad, es de decisiva importancia para muchos fenómenos observados en su superficie. En especial, la zona del ecuador gira mucho más rápidamente que la zona de los polos. Teniendo en cuenta que el periodo de rotación del Sol, observado desde estrellas fijas, es de unos 27 días con 9 horas (valor medio, a $\pm 16°$ de latitud), y que las estrellas enanas (con una masa del orden de la del Sol, pero casi cuatro veces más pequeñas que éste) tienen un periodo de rotación del orden de horas, nos parece casi inimaginable que en las estrellas de neutrones (con masas de 1,4 a $2-3$ masas solares, y radios de $10-20$

km) el periodo de rotación sea del orden de segundos, o incluso de milisegundos. La estrella de neutrones actúa como un enorme imán, con el eje del campo magnético inclinado un ángulo con respecto al eje de rotación, y, así, emitiendo radiación electromagnética desde los polos magnéticos. La fuente de energía del campo magnético es la *energía de rotación* de la estrella. Las estrellas de neutrones en rotación muy rápida y altamente magnetizadas son llamadas púlsares, debido a que cuando el rayo de radio que éstas emiten (desde sus polos magnéticos) en su barrido (con la rotación de la estrella) apunta hacia la Tierra —a intervalos regulares, como un efecto faro— detectamos "pulsos" radio, a intervalos típicamente de 0,25 a 2 segundos; aunque también se han detectado púlsares con periodos de pulsación del orden de 1 a 10 milisegundos.

Simulaciones dinámicas de modelos estelares construidos numéricamente muestran que es muy probable que estrellas de neutrones formadas recientemente en la coalescencia de estrellas de neutrones binarias (o en colapsos supernova) estén rotando diferenciablemente. Y, por la masa que se estima tienen las estrellas de neutrones en binario, se podría concluir que su fusión lleva a un colapso inmediato formando un agujero negro. Sin embargo, se sabe que tanto la presión térmica como la rotación —sobre todo la rotación diferencial— pueden aumentar la masa máxima permitida; en estrellas que están rotando de forma diferencial el núcleo puede rotar más rápido que la envoltura, de forma que el núcleo soporta rotación rápida antes de que la masa "shedding" (o de desprendimiento) se alcance en el ecuador.

El espacio-tiempo de estrellas rotantes *en equilibrio*, esto es, en rotación estacionaria, y con campos gravitatorios que se suponen muy fuertes, es descrito por las ecuaciones de campo gravitatorio relativistas —ecuaciones de campo de Einstein— estacionarias y axisimétricas para fluido perfecto (como fuente) en el interior, acotado, con un exterior vacío asintóticamente plano. Como es muy difícil resolver estas ecuaciones de estructura estelar en rotación —de hecho, por el momento todavía no se ha encontrado de forma exacta ninguna solución global (interior y exterior pegados) y no degenerada—, se analizan las mismas ecuaciones para determinar propiedades generales que cualquier solución particular de las ecuaciones ha de poseer.

Por otra parte, debido a la alta complejidad de las ecuaciones, es común suponer rotación rígida en el modelo estelar con el fin de simplificar, no solamente para la obtención de soluciones, si no también en pruebas de existencia o búsqueda de propiedades generales. En un trabajo muy interesante [1] se han derivado cotas para el radio ecuatorial y para una combinación de masa, momento angular, volumen y radio de un modelo estelar con rotación rígida.

Sin embargo, como se supone que la gran mayoría de objetos compactos en el universo tienen rotación diferencial, nos parece interesante intentar obtener propiedades generales de estrellas que están rotando de forma estacionaria y diferencial (como decíamos antes, intentando extraer de la teoría —analizando las ecuaciones— propiedades que las funciones potenciales o métricas que describen el modelo estelar han de satisfacer). Éste es precisamente el objetivo de la primera parte de esta disertación (Capítulo 2), donde derivaremos ciertas propiedades generales de modelos relativistas estelares con rotación diferencial. En la segunda parte (Capítulo 3) estudiaremos con detalle propiedades de dos soluciones exactas interiores con rotación rígida.

Introducción al Capítulo 2

Uno de los efectos relativistas más intrigantes producidos por la rotación de una estrella es el efecto arrastre (o "dragging") de campos inerciales. En la teoría newtoniana la presencia de un cuerpo masivo no afecta a la determinación de un campo o sistema de referencia inercial. Sin embargo, en el marco de la teoría de la relatividad general, un cuerpo masivo en rotación tiende a arrastrar los campos inerciales con él: consideremos una partícula prueba en reposo (que no rota) con respecto a observadores cercanos, y situada en el campo gravitatorio de una estrella relativista en rotación; un observador situado muy lejos de la estrella, esto es, en el infinito espacial, y que no rota, vería sin embargo que la partícula prueba u observador inercial (localmente en reposo) está rotando (sobre su propio eje) con una *velocidad angular* llamada *de arrastre cumulativo*, o simplemente, *velocidad de arrastre.* [2, 3]

Físicamente se espera que este arrastre de los campos inerciales ocurra en la misma dirección que la rotación de la estrella. También, al menos en el caso de rotación rígida, o sea, cuando el fluido de la estrella gira como un todo (perfil de rotación plano), se espera que si la velocidad angular del fluido (constante) es positiva

—digamos, en la dirección de la coordenada axial ϕ—, entonces la velocidad de arrastre sea menor que la velocidad angular del fluido.

De hecho, Lindblom (1978) [4] e independientemente Hansen y Winicour [5] parecen demostrar estos resultados; sin embargo, la prueba que dan no satisface explícitamente los requerimientos correspondientes al aplicar el teorema de Hopf (un principio de máximo) en lo concerniente a la regularidad y no divergencia.

Una acotación de la velocidad de arrastre por la velocidad angular del fluido en cada punto interior de éste —ya en el caso general de rotación diferencial— es interesante, pues determina la positividad de la *densidad de momento angular* suponiendo que la ecuación de estado (barotrópica) del fluido estelar satisface la condición de energía débil.

En el caso general de rotación diferencial, sin embargo, tal relación no se espera en principio que sea tan simple; primero, porque diferentes partes del fluido podrían girar en direcciones opuestas sobre el mismo eje, y, segundo, porque el perfil de rotación (distribución de velocidad angular) del fluido no se puede prescribir libremente, sino que es determinado por las ecuaciones de campo gravitatorio, en concreto por una condición de integrabilidad de éstas, dada por la ecuación de Euler, a través de una *ley de rotación*.

Hansen y Winicour (1977) [5] han intentado dar un resultado (en rotación diferencial), sin embargo, necesitan suponer que el fluido de la estrella ocupa todo el espacio, hipótesis no física en un modelo estelar.

Así pues, uno de los objetivos del presente trabajo es encontrar condiciones generales y físicamente razonables sobre la ley de rotación (determinando el modelo de rotación diferencial) que garanticen la positividad de la densidad de momento angular; por ejemplo, a través de la acotación de la velocidad de arrastre por la velocidad angular del fluido.

De forma más general (independientemente de la ley de rotación), y como Thorne conjetura [2], es de esperar que el “valor medio” [con respecto a una función densidad] de la velocidad de arrastre sea menor que el valor medio de la velocidad angular del fluido. Sin embargo, no conocíamos en la literatura ninguna prueba de un resultado en esta dirección, conformando esto otro propósito del presente trabajo.

Introducción al Capítulo 3

El análisis de propiedades *de una solución particular* de las ecuaciones de campo de Einstein que representa el espacio-tiempo de una estrella relativista en rotación estacionaria —y, especialmente, la comparación con resultados conocidos en el dominio newtoniano— puede mejorar nuestra intuición dentro del régimen relativista.

Desafortunadamente no se ha encontrado hasta ahora ninguna solución *exacta* en relatividad general representando el campo gravitatorio *global* (ambos exterior e interior) generado por una masa de fluido perfecto, autogravitante, axisimétrico y en rotación estacionaria, con un interior no-vacío en el sentido topológico. [Aunque las condiciones de pegado para una superficie temporal están bien establecidas, el pegado interior-exterior de soluciones estacionarias y axisimétricas genéricas es un problema que todavía no ha sido matemáticamente resuelto.] Debido a esta falta de una solución global exacta no-degenerada, el análisis de propiedades de cualquier resultado parcial nos parece interesante.

Existen en la literatura numerosos tratamientos basados en la integración numérica de las ecuaciones de campo, o en esquemas de aproximación válidos para velocidades de rotación pequeñas (en particular, aplicados al cálculo de la forma de la superficie del fluido, dirección del eje de rotación local, y topología de las figuras). Encontramos asimismo algunos resultados exactos sobre la familia de soluciones (exteriores) de Kerr que describe configuraciones de agujeros negros estacionarios (para los que se analiza el significado de "fuerzas centrífugas"). Sin embargo, sorprendentemente, los resultados basados en las halladas soluciones exactas interiores (para ambos casos, rotación rígida y diferencial) que encontramos eran mínimos.

Se conocen muy pocas soluciones interiores de fluido perfecto, en rotación estacionaria, axisimétricas, con un borde finito de presión nula (superficie del fluido), satisfaciendo condiciones de energía positiva, y poseyendo no más campos de Killing que los dos asociados a las simetrías estacionaria y axial. (Para una revisión, véase p. ej. [6].) Entre estas soluciones hay una obtenida por Wahlquist [7], como un caso especial de una familia más grande de soluciones, y otra obtenida por Kramer [8]. Hasta ahora, ninguna de estas dos soluciones ha sido pegada a una solución exterior de vacío para una configuración en rotación. Para la solución de Wahlquist se ha intentado demostrar que tal solución exterior asintóticamente plana no existe [9], aunque ésta no es ninguna prueba definitiva.

La solución de Wahlquist se conoce desde 1968 y, sin embargo, nada se sabía sobre sus propiedades físicas. La solución de Kramer puede ser obtenida como un límite de esta solución de Wahlquist en el parámetro de rotación.

En la segunda parte de esta disertación se obtienen propiedades geométricas, cinemáticas y dinámicas de estas dos soluciones exactas interiores (fluido perfecto) de Wahlquist y de Kramer.

Capítulo 2

Estrellas relativistas en rotación diferencial: cotas en la velocidad de arrastre y en la energía de rotación

Nos proponemos primeramente dar una prueba correcta y transparente de los resultados previamente citados [4, 5]: en concreto, demostrar (en el caso general de rotación diferencial) que si la distribución de velocidad angular de la estrella tiene signo, digamos "positivo", esto es, si todas las partículas del fluido giran en un mismo sentido, entonces también la velocidad de arrastre es positiva; y demostrar que en el caso de rotación rígida con velocidad angular (constante) positiva, la densidad de momento angular es positiva.

Por otra parte, lo que de verdad interesa es estudiar la positividad de la velocidad de arrastre y de la densidad de momento angular, desde la ley de rotación —en el caso general de rotación diferencial, donde el perfil de rotación no se puede elegir libremente sino que se obtiene junto con las funciones métricas como soluciones de las ecuaciones de campo, una vez dada una ley de rotación compatible.

Consideraremos una solución de las ecuaciones de campo de Einstein estacionarias y axisimétricas para fluido perfecto, con vacío en el exterior, donde la solución es asintóticamente plana, que representa el espacio-tiempo de una estrella relativista en rotación estacionaria. Si además suponemos que el movimiento del fluido estelar es puramente rotacional (o sea, libre de convección), entonces podemos elegir coordenadas adaptadas a las simetrías de forma que las (seis) ecuaciones de campo se reducen a (c.f. [10]) cuatro ecuaciones elípticas y semilineales (en las respectivas cuatro funciones potenciales, o funciones métricas) más la ecuación de Euler, que aquí consideraremos para rotación diferencial.

Especialmente la componente $(t\,\phi)$ es una ecuación elíptica y lineal en el potencial velocidad de arrastre. El único problema al escribir las ecuaciones en estas coordenadas es la singularidad de coordenadas en el eje de rotación; sin embargo, haciendo uso de la simetría axial del problema, un *levantamiento* al espacio $\mathbb{R}^n$ plano (como herramienta puramente matemática, sin significado físico) evita esta singularidad de coordenadas, [10]. En particular, a la ecuación en la velocidad de arrastre corresponde $n = 5$. Así dicha ecuación se escribe de una forma regular y con coeficientes medibles y acotados. Esto nos permite aplicar principios de máximo (mínimo) a varias desigualdades diferenciales derivadas de esta ecuación para sub-(super-)soluciones *generalizadas* que, haciendo uso del pegado C^1 en la superficie de la estrella (de las soluciones interior y exterior) y de la planitud asintótica de la métrica en el infinito espacial, nos lleva a las acotaciones que estábamos buscando; encontramos así una prueba clara y rigurosa de que (1) el arrastre ocurre en la misma dirección que la rotación de la estrella, de que (2) la densidad de momento angular de un modelo estelar en rotación rígida tiene el mismo signo que la velocidad angular del fluido, y (3), en el caso general de rotación diferencial, se demuestra que para una clase de leyes de rotación (muy general, compatible con las ecuaciones de campo, y físicamente relevante) la velocidad de arrastre tiene signo, y está acotada en cada punto interior por la velocidad angular del fluido (entonces con el mismo signo), determinando así el signo de la densidad de momento angular. Éstas y otras acotaciones (también en el exterior) constituyen gran parte (Sec. IV) de la primera de las publicaciones aquí transcritas, [11], a la que nos referiremos de ahora en adelante como **Publicación I**.

Otro objetivo era encontrar una acotación general, independiente de la ley de rotación, en forma de desigualdad de “valores medios” (de velocidad de arrastre y velocidad angular del fluido). A tal efecto estudié primeramente el caso de rotación *lenta* y diferencial, esto es, donde en un desarollo de las ecuaciones de campo de Einstein en potencias del parámetro de rotación (o de velocidad angular del fluido) se consideran las perturbaciones (que serían soluciones aproximadas, obtenidas numéricamente) reteniendo sólo los términos de primer y segundo orden. Sin embargo, dado que un desarrollo del potencial velocidad de arrastre en potencias del parámetro de velocidad angular sólo contiene potencias impares, al calcular efectos a segundo orden en lo referente a la velocidad de arrastre (o cantidades lineales en ella), basta con incluir las correcciones lineales (primer orden) en ésta. De hecho, la única perturbación de primer orden traída por la rotación es el arrastre de los campos inerciales; la estrella es todavía esférica, porque las funciones potenciales que deforman la estrella son de segundo orden.

En este caso, esto es, a primer orden en la velocidad de arrastre, resulta que las ecuaciones de campo todavía no restringen el perfil de rotación (a través de una ley de rotación dada). En consecuencia, el resultado que obtenemos en el caso general (**Publicación I**, Prop. 3) previamente comentado, relativo en particular a la positividad de la densidad de momento angular, no es aplicable al caso de rotación lenta. Sin embargo, la linealidad (en la velocidad angular del fluido) de la ecuación para el potencial de arrastre de una configuración en rotación lenta permite que la positividad de la densidad de momento angular esté garantizada siempre que la amplitud del perfil de rotación esté acotada de un cierta forma. Este refinamiento de las cotas obtenidas en la velocidad de arrastre en el límite de rotación lenta constituye la primera parte [propiedades (*b*) y (*c*) en Sec. V] de la segunda de las publicaciones aquí transcritas, **Publicación II**, con referencia [12].

Además, desde un punto de vista matemático, en este artículo se muestra cómo pueden ser tratadas las singularidades de la subyacente ecuación en derivadas parciales que ocurren en el eje de rotación.

En un resultado no publicado, que incluimos aquí tras **Publicación II**, se estudia el comportamiento cualitativo del potencial de arrastre —o distribución de velocidad angular de arrastre cumulativo— en función de la distribución de velocidad angular del fluido, para una configuración con rotación lenta y diferencial.

Sin embargo, como ya apuntábamos previamente, el motivo de estudiar configuraciones de rotación lenta era buscar una desigualdad de valores medios de velocidad de arrastre y velocidad angular del fluido. En efecto, considerando los desarrollos de ambas funciones en armónicos esféricos —en concreto, en el sistema ortogonal de los polinomios de Jacobi, derivadas de los polinomios de Legendre—, los coeficientes (de Jacobi) de primer orden (de variable radial r) de la velocidad de arrastre satisfacen una ecuación lineal [proveniente de la ecuación $(t\,\phi)$ general] cuyo producto por el coeficiente de la velocidad de arrastre, integrado (de $r = 0$ a $r = \infty$) por partes, y haciendo uso del comportamiento asintótico de los coeficientes velocidad de arrastre, lleva a una desigualdad integral. Con un sencillo cálculo de álgebra lineal, esta última implicará una desigualdad entre dos integrales que se pueden ver como valores medios (respecto a una función densidad) de la velocidad de arrastre y de la velocidad angular del fluido. [Aunque la diferencia de estas integrales es a segundo orden en el parámetro de rotación, su signo depende sólo de cantidades a primer orden.]

Además, esta desigualdad conduce a la positividad de la así llamada *energía total de rotación* [13] —diferencia en masa-energía total entre una estrella relativista en rotación lenta y diferencial, y una no-rotante con el mismo número de bariones y la misma distribución de entropía— y a una cota superior de ésta, proporcionando así una prueba alternativa (y mucho más simple) a la dada por Hartle en [13] de estas cotas [propiedad (d) en Sec. V de **Publicación II**].

Lo interesante es que esta nueva prueba se puede generalizar al caso de rotación diferencial fuera del límite de rotación lenta, como muestra la segunda parte del primer artículo, Sec. V de **Publicación I**: la ecuación para la velocidad de arrastre [componente $(t\,\phi)$] se puede escribir en forma de divergencia; luego, tras multiplicarla por la velocidad de arrastre, aplicar el teorema de Gauss, y utilizar el comportamiento asintótico de las funciones métricas, obtenemos la desigualdad integral que buscábamos. El resto es leer ésta de una forma adecuada. Por una parte nos lleva a que el valor medio (con respecto a una densidad intrínseca) de la velocidad de arrastre es menor que el valor medio de la velocidad angular del fluido [sin ninguna restricción que concierna a la ley de rotación, es decir, completamente general], confirmando así lo conjeturado por Thorne [2]. Y por otra parte implica asi mismo la positividad y una cota superior de la energía total de rotación (con frecuencia objeto de análisis en tratamientos numéricos), generalizando así el resultado dado por Hartle [13] en el límite de rotación lenta (y diferencial) a un régimen de rotación general diferencial.

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Relativistic stars in differential rotation: bounds on the dragging rate and on the rotational energy

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Abstract

For general relativistic equilibrium stellar models (stationary axisymmetric asymptotically flat and convection-free) with differential rotation, it is shown that for a wide class of rotation laws the distribution of angular velocity of the fluid has a sign, say "positive", and then both the dragging rate and the angular momentum density are positive. In addition, the "mean value" (with respect to an intrinsic density) of the dragging rate is shown to be less than the mean value of the fluid angular velocity (in full general, without having to restrict the rotation law, nor the uniformity in sign of the fluid angular velocity); this inequality yields the positivity and an upper bound of the total rotational energy.

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I. INTRODUCTION

One of the most interesting of the relativistic effects produced by the rotation of a star is the dragging of inertial frames (also called Lense-Thirring effect).[1] This has been classically described in terms of its *local* effects on gyroscopes and particles. However, its *cumulative* effects on the motion of particles give a much simpler description: a locally non-rotating test particle, that is dragged along in the gravitational field of the star, has an angular velocity, as seen from a non-rotating observer at spatial infinity, which is named *angular velocity of cumulative dragging*, or *rate of rotational dragging*, shortly called *dragging rate*.[2,3] It is physically expected that, for isolated rotating stars in thermodynamic equilibrium, this dragging rate A has the same sign (rotation sense) as the fluid angular velocity Ω, if this one has a "uniform" sign throughout the fluid (in the general differentially rotating case). Indeed, Lindblom[4] and, independently, Hansen and Winicour (1977)[5] seem to establish this result, however, without explicitly fulfilling the corresponding requirements when applying the Hopf theorem (a maximum principle) to an elliptic operator in a certain domain, concerning the boundedness of its coefficients on the boundary of the domain, specifically on the axis of rotation, and the C^1 (and not C^2) regularity of the metric functions across the surface of the star.

Also, (assuming in the description above that the test particle does not collide with the star's matter if it goes through the star) one is tempted to conjecture, in principle in the rigidly rotating case with $\Omega = \text{const.} = \Omega_* > 0$, that the dragging rate is bounded above by the fluid angular velocity, $A \leq \Omega_*$. And Hansen and Winicour (1975)[5] offer some proof of this (although with the same objection as above).

In the general differentially rotating case, however, an analogous relation should not be expected to be so simple; first, because different portions of the star's interior could have opposite rotational motion about the same axis (assuming a convection-free fluid), and, second (even if the fluid angular velocity has a sign), because, due to the integrability condition of the equation of motion, the distribution of fluid angular velocity, Ω-profile, cannot be freely prescribed; instead, it is derived (together with the potential functions, integrating the field equations) once an appropriate *rotation law* is given. Most of the literature concerning numerical works on differentially rotating neutron stars make generally the ansatz for a certain rotation law which yields $A \leq \Omega$. Nevertheless, for a more general law such a relation is not so obvious. Hansen and Winicour (1977)[5] have made some attempts to give a result, however they needed the unphysical assumption that the star's matter occupies the whole space.

One of the aims of this work is precisely to find general and physically reasonable assumptions on the rotation law of a differentially rotating stellar model, so that the dragging rate is (at each interior point) less than the fluid angular velocity, and, hence, the *angular momentum density* is positive (vanishing on the axis) provided the weak energy condition is satisfied. For that matter we consider the time-angle field equation's component, which is elliptic and linear in the dragging potential A in coordinates adapted to the symmetries. The approach with the metric in these coordinates is attractive because the field equations become semilinear elliptic. Specially, they reduce to four (coupled) elliptic equations for the four metric functions. One has however to control the coordinate singularities of the equations on the axis of rotation, but these can be treated mathematically using the axial symmetry of the physical problem.[6] So handled, the elliptic equation in A writes in a "regular" form and has bounded coefficients; this allows us to apply a maximum principle to several differential inequalities, which, using the C^1-matching on the star's surface and the asymptotic flatness of the metric, will lead to the mentioned and other interesting inequalities.

More generally, as was conjectured by Thorne[2] (p. 245), the *mean value* of A is expected to be less than the *mean value* of Ω. However, to my knowledge, there is in the literature no explicit and so general result in this direction. The other purpose of the present work is then to derive a "general" inequality on "mean values" with respect to a density function. In addition, related to this question is the concept of *total rotational energy* of the star. Hartle[7] has given bounds for this rotational energy in the slow rotation limit, which we aim here to generalize.

The paper is organized as follows. After reviewing in Sec. II the model for a relativistic star which is rotating differentially, Sec. III is devoted to handle the concerned field equation, elliptic and linear in the dragging potential, with special attention to the regularity and boundedness properties of the involved functions, as a preparation allowing us to apply the maximum principles (reviewed in the Appendixes) in Sec. IV, where inequalities concerning mainly the positivity of the dragging rate and of the angular momentum density are derived. In Sec. V a general "mean values inequality" is derived in full general; and the positivity and upper bound of the total rotational energy is established in the general differentially rotating case. Finally, in Sec. VI the relevant results are briefly summarized.

II. MODEL FOR A DIFFERENTIALLY ROTATING RELATIVISTIC STAR

The spacetime of an isolated rotating star in thermodynamic equilibrium within general relativity theory is generally represented by an asymptotically flat stationary axisymmetric 4-dimensional Lorentzian manifold $(\mathcal{M}, \boldsymbol{g})$, with metric $\boldsymbol{g} = g_{\alpha\beta}dx^\alpha dx^\beta$ satisfying Einstein's equations,

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}R\, g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}\,, \tag{1}$$

for the energy-momentum tensor of a perfect fluid, $T_{\alpha\beta} = (\varepsilon + p)\, u_\alpha u_\beta + p\, g_{\alpha\beta}$, with 4-velocity u^α, energy density ε, and pressure p. Signature of $\boldsymbol{g}$ is here considered to be $(-+++)$. Since the star is isolated, the matter (perfect fluid) is confined in a compact region in the space (interior), with vacuum, $T_{\alpha\beta} = 0$, on the outside.

We denote the two (commuting) global time and axial Killing vector fields[8] by $\boldsymbol{\xi} = \partial_t$ and $\boldsymbol{\eta} = \partial_\phi$, respectively, where $x^0 \equiv t$ labels the space-like hypersurfaces which are invariant under time translations, and $x^1 \equiv \phi$ is the axial-angle coordinate around the axis of rotation, given by $\boldsymbol{\eta} \equiv 0$; $\quad (t, \phi) \in \mathbb{R} \times [0, 2\pi[$. The metric components will then only depend on the two remaining spatial coordinates, $g_{\alpha\beta} = g_{\alpha\beta}(x^2, x^3)$.

We shall assume that the fluid motion is purely azimuthal (non-convective), i.e. the fluid 4-velocity is contained in the 2-surface spanned by the two Killing fields, (as 1-forms)

$$\mathbf{u} \wedge \boldsymbol{\xi} \wedge \boldsymbol{\eta} = 0 \qquad\qquad \text{(circularity condition)}\,. \tag{2}$$

In that case it can be seen[9] that the 2-surface elements orthogonal to the 2-dimensional group orbits of the Killing fields are surface forming (the same holds in the vacuum region); and, consequently, the metric may be written in a form which is explicitly symmetric under the change $(t, \phi) \to (-t, -\phi)$. In the 2-surfaces orthogonal to the orbits we can always introduce *isotropic coordinates* $(x^2, x^3) = (\rho, z)$ without loss of generality, so that the metric can always be reduced to the standard form[3,10]

$$\boldsymbol{g} = g_{\alpha\beta}dx^\alpha dx^\beta = -e^{2U}dt^2 + e^{-2U}\left[\rho^2 e^{2B}(d\phi - A\, dt)^2 + e^{2K}(d\rho^2 + dz^2)\right]\,, \tag{3}$$

where the metric functions K, U, B, and A depend only on the (ρ, z)-coordinates of the "meridian plane". Here ρ and z are cylindrical coordinates at the asymptotically flat infinity, and, using the remaining freedom of conformal transformations in the meridian plane, we choose these coordinates such that $\rho = 0$ represents the axis of rotation and $(\rho, z) \in \mathbb{R}_0^+ \times \mathbb{R}$ (denoting $\mathbb{R}_0^+ := \{\mathrm{x} \in \mathbb{R} \,|\, \mathrm{x} \geq 0\}$). The metric functions $\rho\, e^B$, U, and A can be written as invariant combinations of the Killing fields in the form

$$
\begin{aligned}
\rho^2 e^{2B} &= -\det((g_{\mu\nu})_{\mu,\nu=t,\phi}) = -\boldsymbol{g}(\boldsymbol{\xi},\boldsymbol{\xi})\,\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{\xi},\boldsymbol{\eta})^2 \\
e^{2U} &= \frac{\rho^2 e^{2B}}{\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{\eta})} \\
A &= -\frac{\boldsymbol{g}(\boldsymbol{\xi},\boldsymbol{\eta})}{\boldsymbol{g}(\boldsymbol{\eta},\boldsymbol{\eta})}\,,
\end{aligned}
\tag{4}
$$

and they can be interpreted physically as follows: $\rho\, e^B$ represents a sort of distance from the rotation axis (and, hence, B is, to some extent, a measure how far is ρ from being that distance); U is a generalization of the gravitational potential; and A is the *angular velocity of cumulative dragging*, or *dragging rate.* The remaining metric function is K, the conformal factor in the meridian plane.

Throughout the following we shall denote the closure and the boundary of a set X by $\overline{X}$ and ∂X, respectively. We fix the notions

$$
\begin{aligned}
I &\equiv \text{interior of the star} := \{(\rho,z) \in \mathbb{R}_0^+ \times \mathbb{R} \mid p(\rho,z) > 0\} \subset \mathbb{R}_0^+ \times \mathbb{R} \\
E &\equiv \text{exterior of the star} := (\mathbb{R}_0^+ \times \mathbb{R}) \setminus \overline{I} \subset \mathbb{R}_0^+ \times \mathbb{R} \\
S &\equiv \text{star's surface} := \overline{I} \cap \overline{E} = \partial I \subset \mathbb{R}_0^+ \times \mathbb{R}\,,
\end{aligned}
\tag{5}
$$

I and E open in the *induced topology in* $\mathbb{R}_0^+ \times \mathbb{R} \subset \mathbb{R}^2$; that means, although part of the axis ($\rho = 0$) is in I (and part in E), the only points of the axis which are in $\partial I = S$ (and in $\partial E = S \cup \{\infty\}$) are the poles, if they exist. The set $I \subset \mathbb{R}_0^+ \times \mathbb{R}$ is supposed to be bounded and connected. Concerning the regularity of $S = \partial I$, we assume it satisfies an exterior sphere condition everywhere (cf. Definition in Appendix A).

Within our star model, the *matching* conditions (from the interior and the exterior solutions) require that the pressure vanishes identically on the star's surface, $p = 0$ on S. In the exterior ($T_{\alpha\beta} = 0$) we extend definitions such that $\varepsilon = p = 0$ on

E. Furthermore, ε and p satisfy a barotropic equation of state in the interior,

$$\varepsilon = \varepsilon(p) \quad \text{in } \overline{I}\,. \tag{6}$$

We assume the pressure p to be continuous with respect to the coordinates, and also $p \mapsto \varepsilon(p)$ a continuous function,

$$p \in C^0(\mathbb{R}_0^+ \times \mathbb{R})\,, \qquad p \mapsto \varepsilon(p) \ \in C^0(\mathbb{R}_0^+)\,, \tag{7}$$

satisfying the *weak energy condition*,[11]

$$\varepsilon + p \geq 0 \qquad (\text{in } \mathbb{R}_0^+ \times \mathbb{R})\,. \tag{8}$$

[Notice, by the definition of the interior, (5), if $\varepsilon \geq 0$ in $\overline{I}$, as it is generally assumed, we shall have even $\varepsilon + p > 0$ in I, and, hence, condition (8) follows. In addition, since the equation of state is defined only in the interior, (6), requirement (7) does not guarantee the continuity of ε across the star's surface (where $p = 0$), namely, if $\varepsilon(p = 0) > 0$, then a jump discontinuity of ε across the star's surface occurs.]

From the circularity condition (2) on the fluid 4-velocity (in $\overline{I}$), this is of the form

$$\mathbf{u} = u^t(\boldsymbol{\xi} + \Omega\boldsymbol{\eta}), \qquad \text{where } \ \Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}$$

is the angular velocity of the fluid measured by a distant observer in an asymptotically flat spacetime, and the fact that the 4-velocity $\mathbf{u}$ is a unit time-like vector field determines the normalization factor u^t, such that $\boldsymbol{g}(\mathbf{u},\mathbf{u}) = -1$, i.e.

$$(u^t)^{-2} = e^{2U} - \rho^2 e^{2(B-U)}(\Omega - A)^2 \ =: \ N\,, \tag{9}$$

from where $N = (u^t)^{-2} > 0$ in $\overline{I}$. Indeed, we do not allow that the velocity of light is approached somewhere, and, hence, even

$$N \geq \text{const.} > 0 \quad \text{in } \overline{I}. \tag{10}$$

We consider a star rotating differentially with a distribution of angular velocity (*rotation profile*) $\Omega = \Omega(\rho, z)$, a continuously differentiable function,

$$\Omega \in C^1\left(\overline{I}\right)\,. \tag{11}$$

However, the Ω-profile of the fluid cannot be freely chosen, this shows up in the following. The integrability conditions of the field equations (1), that is, the equation of hydrostatic equilibrium ${T^{\alpha\beta}}_{;\beta} = 0$ (from ${G^{\alpha\beta}}_{;\beta} = 0$) (where ';' denotes covariant derivative), particularly, its part orthogonal to the fluid 4-velocity $\mathbf{u}$, reduces to the Euler equation,

$$dp = -(\varepsilon + p)\,\mathbf{a}\,, \tag{12}$$

where $\mathbf{a}$ is the 4-acceleration of the fluid, $\mathbf{a} = \nabla_{\mathbf{u}}\mathbf{u}$. Specifically,

$$\mathbf{a} = dV + u^t u_\phi\, d\Omega\,, \qquad V \equiv \frac{1}{2}\ln N\,, \tag{13}$$

$u^t u_\phi = \rho^2 e^{2(B-U)}(\Omega - A)N^{-1}$. But the integrability condition of Eq. (12) taking into account (6) is $d\mathbf{a} = 0$; following then, from (13), $d(u^t u_\phi)\wedge d\Omega = 0$. The special case where $\Omega =$ const. is called *rigid rotation* (or *uniform rotation*). In general we shall have $\Omega \neq$ const., following then,

$$u^t u_\phi = \mathcal{F}(\Omega)\,, \tag{14}$$

for some function $\mathcal{F}$, *rotation law.* By specifying the function $\mathcal{F}(\Omega)$ a specific model of *differential rotation* is obtained. [Note, since in the Newtonian limit $u^t u_\phi \to \rho^2\,\Omega$, Eq. (14) expresses the general relativistic generalization of the Newtonian "rotation on cylinders" theorem, $\Omega = \mathcal{G}(\rho^2)$].

Further requirements on our stellar model are:

a. the metric functions are (at least) two times continuously differentiable in the interior and in the exterior of the star, and continuously differentiable everywhere (cf. Note in Sec. III B),

$$K, U, B, A \in C^2(I)\cap C^2(E)\cap C^1(\mathbb{R}_0^+\times\mathbb{R})\,; \tag{15}$$

b. in order that the metric functions are symmetric with respect to the z-axis ($\rho = 0$) ("axisymmetric solutions"), and, hence, the metric (3), defined on $\mathcal{M}$ excluding the axis, can be extended to an at least C^1 axisymmetric tensor field in the whole spacetime $\mathcal{M}$, we assume that

$$\text{as } \rho \to 0, \qquad \partial_\rho K,\ \partial_\rho U,\ \partial_\rho B,\ \partial_\rho A \to 0\,, \tag{16}$$

and, for completeness, also $\partial_\rho\varepsilon,\ \partial_\rho p \to 0$;

c. finally, by the asymptotic flatness requirement, denoting $\mathcal{D} := (\partial_\rho, \partial_z)$,

$$\text{as } R := (\rho^2 + z^2)^{1/2} \to \infty, \quad K, U, B, A \to 0 \quad \text{and} \quad \mathcal{D}K,\ \mathcal{D}U,\ \mathcal{D}B,\ \mathcal{D}A \to \mathbf{0}\,. \tag{17}$$

Notice, from C^1 regularity, in (15), and asymptotic flatness, (17), it follows, in particular, that the metric functions and their derivatives are bounded,[12]

$$|K|, |U|, |B|, |A| < \infty \quad \text{and} \quad \|\mathcal{D}K\|, \|\mathcal{D}U\|, \|\mathcal{D}B\|, \|\mathcal{D}A\| < \infty \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}. \quad (18)$$

III. THE ELLIPTIC EQUATION FOR THE DRAGGING RATE

A. The time-angle field equation component

The $(t\phi)$ component of Einstein's equation (1) in these coordinates takes the form[6]

$$\partial_{\rho\rho}A + \partial_{zz}A + \frac{3}{\rho}\partial_\rho A + \langle 3\mathcal{D}B - 4\mathcal{D}U\,,\,\mathcal{D}A\rangle = -\psi^2 \cdot (\Omega - A)\,, \quad (19)$$

$$\text{with} \qquad \psi^2 := 16\pi \frac{e^{2K}}{N}(\varepsilon + p) \quad [\geq 0, \quad \text{by (8) and (10)}]\,, \quad (20)$$

where $\langle .,. \rangle$ denotes the Euclidean scalar product. Since, from condition (16), $v(\rho, z) = v(-\rho, z)$ for $v = K, U, B, A$, i.e. we are considering only axisymmetric solutions of the field equations, and since only "axisymmetric operations" appear in these equations, we consider the following transformation (in the spirit of Ref. 6) in order to avoid the coordinate singularity (of Eq. (19)) on the axis of symmetry (z-axis, i.e. $\rho = 0$). To this end we use the 5-*lift of* each function $v \equiv v(\rho, z)$ (on $\mathbb{R}^5$), for the metric functions $v = K, U, B, A$ and also for $v = \Omega, \varepsilon, p$, where the *$n$-lift of* $v : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ on flat $\mathbb{R}^n$, axisymmetric around the x_n-axis, is defined as follows

$$v \mapsto \tilde{v} \quad \text{such that} \quad \tilde{v}(x) \equiv \tilde{v}(x_1, \ldots, x_n) := v\big(\rho = ({x_1}^2 + \cdots + x_{n-1}^2)^{1/2}\,,\, z = x_n\big)\,, \quad (21)$$

and, for every function $\tilde{v} : \mathbb{R}^n \to \mathbb{R}$, the *meridional cut (in direction x_1) of* $\tilde{v}$,

$$\tilde{v} \mapsto v \qquad \text{such that} \qquad v(\rho, z) := \tilde{v}(\rho, 0, \ldots, 0, z)\,. \quad (22)$$

For axisymmetric functions, both operations are isomorphisms and inverse to each other; but the relevant properties of n-lift and meridional cut are that (a) they leave the regularity conditions and the norms invariant, (b) they commute with "axisymmetric operations", in particular, with all operations in Eq. (19), like multiplication and scalar product, yielding especially (for $n = 5$)

$$\langle \mathcal{D}v, \mathcal{D}w \rangle = \langle \nabla\tilde{v}, \nabla\tilde{w} \rangle\,, \quad \text{denoting } \nabla := (\partial_1, \ldots, \partial_5) \qquad (\partial_i \equiv \partial_{x_i}, \quad \partial_{ij} \equiv \partial_{x_i}\partial_{x_j}\ )\,,$$

and, remarkably, (c) they transform the operator $\partial_{\rho\rho} + \partial_{zz} + \frac{n-2}{\rho}\partial_\rho \quad (n \geq 2)$ into the flat n-dimensional Laplacian, and vice versa; having for $n = 5$

$$\partial_{\rho\rho} v + \partial_{zz} v + \frac{3}{\rho}\partial_\rho v = \sum_{i=1}^{5} \partial_{ii}\tilde{v} =: \Delta\tilde{v}\,.$$

Hence, with the 5-lift, Eq. (19) writes in the form

$$\Delta\tilde{A} + \langle 3\nabla\tilde{B} - 4\nabla\tilde{U}\,,\, \nabla\tilde{A}\rangle = -\tilde{\psi}^2 \cdot (\tilde{\Omega} - \tilde{A})\,, \tag{23}$$

$\tilde{\psi}^2$ defined like ψ^2, (20), but with 5-*lifted* functions (on $\mathbb{R}^5$).

B. Regularity and boundedness of the metric functions

Let us see how conditions (15)-(18) transmit through the 5-lift. First, from conditions (15) and (16) it follows

$$\tilde{K}, \tilde{U}, \tilde{B}, \tilde{A} \in C^1(\mathbb{R}^5)\,, \tag{24}$$

because (for $v = K, U, B, A$) $\;v \in C^1(\mathbb{R}_0^+ \times \mathbb{R})\;$ and $\;\partial_\rho v \to 0$ as $\rho \to 0\;$ imply $\tilde{v} \in C^1(\mathbb{R}^5)$.

Note: In fact, as seen in Ref. 6, with the use of these mathematical tools (n-lift and meridional cut, for different numbers n), the elliptic system of field equations (1) may be regarded as a set of *Poisson-like equations*, where the nonlinearities (quadratic terms in the first derivatives of the metric functions) are contained in the inhomogeneous terms on the right hand side. Making the weak requirement that the metric functions and their derivatives are essentially bounded, $\tilde{v}$, $\nabla\tilde{v} \in L^\infty$, since also $\tilde{\varepsilon}$, $\tilde{p} \in L^\infty$ [by condition (7) and $\tilde{\varepsilon} = \tilde{p} = 0$ in the exterior], and $\tilde{\Omega} \in L^\infty$ [by (11)], we have that the right hand side is essentially bounded. Then, by the regularity of Poisson's integral,[13] (at least) $\tilde{v} \in C^{1,\alpha}$ for some $\alpha < 1$; in particular, $\tilde{v} \in C^1$, i.e. (24). This justifies requirements (15) and (16) in Sec. II.

Combining (15) and (16) we obtain also that the 5-lifted metric functions are class C^2 in the interior and in the exterior of the star (in $\mathbb{R}^5$). That is, denoting

$$\mathcal{I} := \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \mid (({x_1}^2 + \cdots + {x_4}^2)^{1/2}\,,\, x_5) \in I\} \subset \mathbb{R}^5 \tag{25}$$

and, analogously, $\mathcal{E}$ and $\mathcal{S}$, from E and S [cf. (5)], respectively, we have for $\tilde{v} = \tilde{K}, \tilde{U}, \tilde{B}, \tilde{A}$,

$$\tilde{v} \in C^2(\mathcal{I}) \cap C^2(\mathcal{E}) \cap C^1(\mathbb{R}^5)\,. \tag{26}$$

The asymptotic flatness condition (17) implies, through the 5-lift, that

$$\text{as} \quad R = \|x\| = ({x_1}^2 + \ldots + {x_5}^2)^{1/2} \to \infty, \qquad \tilde{v} \to 0 \quad \text{and} \quad \nabla\tilde{v} \to \mathbf{0}\,; \tag{27}$$

but $\tilde{v} \in C^1(\mathbb{R}^5)$, that is, $\tilde{v} \in C^0(\mathbb{R}^5)$ and $\nabla\tilde{v} \in [C^0(\mathbb{R}^5)]^5$, yielding, together with conditions (27), their respective boundedness,

$$|\tilde{v}| < \infty \quad \text{and} \quad \|\nabla\tilde{v}\| < \infty \ \text{ in } \mathbb{R}^5\,. \tag{28}$$

C. Notation convention and roundup

We have seen in Sec. III A that Eq. (23) is equivalent to Eq. (19) through the 5-lift and the meridional cut, (21) and (22) for $n = 5$. Furthermore, the 5-lift leaves regularity and boundedness properties invariant; see Sec. III B.

Convention: For simplicity in the notation, we omit throughout the following the symbols '$\tilde{}$' for all 5-*lifted* functions we use. [Once it has been seen how regularity and boundedness properties transmit from the functions defined on $\mathbb{R}_0^+ \times \mathbb{R}$ to the lifted ones (on $\mathbb{R}^5$), and since they are equivalent in terms of positivity, and no explicit reference to the first ones will appear throughout the following section, this notation convention seems appropriate.]

Accordingly, we write Eq. (23) in the form

$$L_0\,A \;=\; -\psi^2 \cdot (\Omega - A)\,, \tag{29}$$

$$\text{with} \qquad L_0\,A := \Delta A + \langle 3\nabla B - 4\nabla U\,,\, \nabla A\rangle\,, \tag{30}$$

$$\begin{aligned} &\psi^2 := 16\pi\frac{e^{2K}}{N}(\varepsilon + p) \ \geq 0 \ \ (= 0 \ \text{ in } \mathcal{E})\,, \\ &\text{and} \quad N := e^{2U} - \rho^2 e^{2(B-U)}(\Omega - A)^2 \ \geq \ \text{const.} > 0 \ \text{ in } \overline{\mathcal{I}}\,, \end{aligned} \tag{31}$$

where $K, U, B, A : \mathbb{R}^5 \to \mathbb{R}$, axisymmetric around the x_5-axis. [Notice, Eq. (29) is so defined in the whole spacetime, interior (fluid) and exterior (vacuum), $\overline{\mathcal{I}} \cup \mathcal{E} = \mathbb{R}^5$, but in the exterior $\psi^2 \equiv 0$ ($\varepsilon = p = 0$) and the vacuum field equation is recovered, $L_0\, A = 0$ in $\mathcal{E}$.] Also, we have (26)-(28), i.e. (with the notation convention)

$$K,\, U,\, B,\, A \in C^2(\mathcal{I}) \cap C^2(\mathcal{E}) \cap C^1(\mathbb{R}^5)\,, \tag{32}$$

$$\text{as} \quad R = \|x\| = ({x_1}^2 + \ldots + {x_5}^2)^{1/2} \to \infty, \qquad K,\, U,\, B,\, A \to 0\,, \tag{33}$$

$$\nabla K,\ \nabla U,\ \nabla B,\ \nabla A \to \mathbf{0}\,, \tag{34}$$

$$|K|,\ |U|,\ |B|,\ |A| < \infty \tag{35}$$

$$\text{and} \qquad \|\nabla K\|,\ \|\nabla U\|,\ \|\nabla B\|,\ \|\nabla A\| < \infty \ \text{ in } \mathbb{R}^5\,. \tag{36}$$

Equation (29), i.e.

$$L\, A := L_0\, A - \psi^2 \cdot A = -\psi^2 \cdot \Omega\,, \tag{37}$$

writes then

$$L\, A \equiv a_{ij}(x)\, \partial_{ij} A + b_i(x)\, \partial_i A + c(x)\, A = g(x)$$

$$\text{with} \qquad \begin{aligned} a_{ij} &\equiv \text{const.} = \delta_{ij} \ \ (= 1 \text{ if } i = j, \text{ and } = 0 \text{ otherwise})\,, \\ b_i &= 3\, \partial_i B - 4\, \partial_i U \qquad (\forall\, i, j \in \{1, \ldots, 5\}), \ \text{ and} \\ c &= -\psi^2 \ \ (\leq 0)\,, \\ g &= c\, \Omega\,, \end{aligned} \tag{38}$$

(where repeated indices indicate summation from 1 to 5). The flat 5-dimensional Laplacian Δ, in (30), ($a_{ij} \equiv \delta_{ij}$), and hence L, is obviously strictly and uniformly elliptic everywhere. The coefficients b_i are measurable and bounded functions everywhere, because B and U are C^1, (32), and have bounded derivatives, (36). On the other hand, for the coefficient c [cf. (31)], since (i) the metric functions are continuous, (32), and bounded, (35); (ii) p is continuous everywhere, (7), and has compact support; (iii) ε is continuous in the (closed) interior $\overline{\mathcal{I}}$, from (7); (iv) Ω is in particular continuous (in $\overline{\mathcal{I}}$), (11), and, hence, measurable; and (v) $N \geq$ const. > 0 (also in $\overline{\mathcal{I}}$), (10); it follows that $c \equiv -\psi^2$ is measurable and bounded in the interior $\mathcal{I}$, and, since $\psi^2 \equiv 0$ ($\varepsilon = p = 0$) in the exterior $\mathcal{E}$, and the boundary (the star's surface) $\partial\mathcal{I} = \mathcal{S}$ is a set of measure zero, we have that the coefficient c is measurable and bounded everywhere. This will allow us in the following section to apply maximum principles in the classical and in the generalized sense to the operator L (and L_0); see Appendixes A and B.

IV. BOUNDS ON THE DRAGGING RATE

A. Positivity of the dragging rate

Proposition 1

If the distribution of angular velocity of the fluid is non-negative (and non-trivial), then the dragging rate is positive everywhere.

$$\Omega \geq 0, \quad \Omega \not\equiv 0 \quad \Longrightarrow \quad A > 0\,.$$

Proof. Consider the domain G defined by a ball in $\mathbb{R}^5$ centered at the origin $x = \mathbf{0}$ and of arbitrarily large radius σ,

$$G := \mathcal{B}_\sigma(\mathbf{0}) \subset \mathbb{R}^5\,. \tag{39}$$

Since A is continuously differentiable in $\mathbb{R}^5$, cf. (32), so is in particular in G; but A and ∇A continuous in $\mathbb{R}^5$ implies that they are 2-integrable (are in L^2) in G; consequently,

$$A \in W^{1,2}(G) \cap C^1(G)\,. \tag{40}$$

Hence, the strictly elliptic linear partial differential equation (in A) with measurable and bounded coefficients, (37), is satisfied in a generalized sense in G; see Appendix B. Remarkably, whenever $\Omega \geq 0$, Eq. (37) yields the differential inequality

$$L\,A \leq 0 \quad \text{in } G\,, \tag{41}$$

i.e. A is a generalized supersolution relative to the operator L and the domain G. We pay now special attention to the behavior of A on the boundary: since the radius of the ball G, σ, is arbitrary, we can make it sufficiently large ($\sigma \to \infty$) such that, by the asymptotic flatness condition on A [cf. (33)], A is arbitrarily small on ∂G,

$$\lim_{\sigma\to\infty} A|_{\partial G} = 0\,. \tag{42}$$

We first observe that $A \not\equiv$ const. [because, by (42) and $A \in C^0(\mathbb{R}^5)$, would be $A \equiv$ const. $= 0$, which yields, by Eq. (37), $\Omega \equiv 0$, and we are assuming $\Omega \not\equiv 0$]. Hence, by the strong minimum principle, Theorem 4 in Appendix B, applied to the differential inequality (41), A cannot attain a non-positive minimum at an interior point of G; using (42), we conclude then $A > 0$ in G, i.e. everywhere. □

Remark 1. A result analogous to Proposition 1 holds with the opposite sense of the rotation; that is, if $\Omega \leq 0$ $(\Omega \not\equiv 0)$, then $A < 0$. This follows because Eq. (29) is invariant with respect to the simultaneous change of sign $(\Omega, A) \to (-\Omega, -A)$.

B. Upper bound Ω. Positivity of the angular momentum density

Hereafter we discuss the sign of the difference $\Omega - A$. Remarkably, this determines the sign of $u^t u_\phi$, which, with assumption (8), is the sign of the *angular momentum density*, integrand of the total angular momentum, given by the "volume" integral[14] $J = \int_{\mathcal{I}} 2\pi \, T_\phi^{\ t} \, (-\mathrm{g})^{1/2} dx$, where $\mathrm{g} \equiv \det(\boldsymbol{g})$ and $T_\phi^{\ t} = (\varepsilon + p)\, u^t u_\phi$.

1. In the rigidly rotating case

Proposition 2

In the particular case of rigid rotation, with $\Omega \equiv$ const. $=: \Omega_* > 0$,

$$0 < A < \Omega_*$$

holds everywhere. As a consequence, in this case, $u^t u_\phi$, *and, hence, the angular momentum density, is non-negative.*

Proof. We consider Eq. (29) for $\Omega \equiv$ const. $= \Omega_* > 0$, i.e. $L_0 \, A = -\psi^2 \cdot (\Omega_* - A)$, which, since the differential operator L_0, (30), is free from linear term, can be rewritten in the form

$$L_0(A - \Omega_*) = -\psi^2 \cdot (\Omega_* - A)\,,$$

or, denoting again the differential operator $L := L_0 - \psi^2$ and defining

$$w(x) := A(x) - \Omega_* \tag{43}$$

in the whole spacetime, $x \in \overline{\mathcal{I}} \cup \mathcal{E} = \mathbb{R}^5$ (as already 5-lifted function; cf. Sec. III A),

$$Lw = L(A - \Omega_*) = L_0(A - \Omega_*) - \psi^2 \cdot (A - \Omega_*) = 0 \quad \text{in } \mathbb{R}^5\,.$$

We have then the strictly elliptic linear (in w) equation

$$Lw = 0 \qquad \text{(in particular) in } G \equiv \mathcal{B}_\sigma(\mathbf{0}) \subset \mathbb{R}^5 \,, \tag{44}$$

where the radius σ is arbitrary, with $w \in W^{1,2}(G) \cap C^1(G)$, by (40) and (43). On the other hand, by the condition of asymptotic flatness on A [in (33)], A is arbitrarily small on ∂G, provided that σ is sufficiently large, i.e. (42); consequently,

$$\lim_{\sigma \to \infty} w|_{\partial G} = -\Omega_* < 0 \,. \tag{45}$$

Since $w \not\equiv$ const. [because, by (45) and continuity, would be $w \equiv \text{const.} = -\Omega_*$ in G, that is, $A \equiv \text{const.} = 0$ in G, which is not allowed, by Eq. (37), since here $\Omega \equiv \Omega_* > 0$], applying the strong maximum principle, Theorem 4 in Appendix B, to Eq. (44), we get that w cannot attain a non-negative maximum at an interior point of G; hence, using (45), $w < 0$ in G (everywhere), i.e. $A < \Omega_*$ everywhere. Moreover, $A > 0$ everywhere, by Proposition 1. This establishes the conclusion of the proposition. □

Observe, in the static case, $\Omega_* = 0$, we would have $Lw = 0$ and $\lim_{\sigma \to \infty} w|_{\partial G} = 0$, following, by the strong maximum and minimum principles, $w \equiv 0$, i.e. $A \equiv \Omega_* = 0$; as expected, $A \equiv 0$.

Remark 2. Likewise, if $\Omega \equiv \text{const.} \equiv \Omega_* < 0$, then $0 > A > \Omega_*$ everywhere, and, hence, the angular momentum density is non-positive. We obtain this by applying Proposition 2 to the function $\hat{A} := -A$, solution of Eq. (29) for $\hat{\Omega}_* := -\Omega_* > 0$ (cf. Remark 1). More explicitly, the angular momentum density of a rigidly rotating stellar model has the same sign as the angular velocity of the fluid. Also, as a result, we have for a fluid rotating rigidly with $\Omega \equiv \text{const.} \equiv \Omega_* \neq 0$

$$0 < |A| < |\Omega_*| \,.$$

2. In the general (differentially rotating) case

In the following we shall assume that a function $\mathcal{F}$ (to be specified) has been given, and we have a solution of the problem, that is, (four) metric functions, K, U, B, and A, and a fluid angular velocity distribution, Ω, satisfying the (four) field equations (1) [in particular, the elliptic equation for A, Eq. (29)] and Eq. (14), $u^t u_\phi = \mathcal{F}(\Omega)$. [Notice, in the interior, where the matter terms do not vanish ($p > 0$, $\varepsilon \geq 0$), substituting into the equation of motion (12) [with (13)] its integrability condition, i.e. Eq. (14), and the equation of state, Eq. (6), we obtain the pressure, p, and the energy density, ε, as functions of ρ, U, B, A, and Ω.]

Remarkably, $u^t u_\phi$ may be written

$$\begin{aligned} u^t u_\phi &\equiv \frac{\rho^2 e^{2(B-U)}(\Omega - A)}{e^{2U} - \rho^2 e^{2(B-U)}(\Omega - A)^2} \\ &= \frac{\varrho^2(\Omega - A)}{1 - \varrho^2(\Omega - A)^2} =: \Phi(\varrho\,,\, \Omega - A) \qquad \text{with} \;\; \varrho := \rho\, e^{B-2U}\,, \end{aligned} \tag{46}$$

where, from (10), $1 - \varrho^2(\Omega - A)^2 = N e^{-2U} \geq$ const. > 0 in $\overline{\mathcal{I}}$. With the defined function (46), Eq. (14) writes

$$\Phi(\varrho\,,\, \Omega - A) = \mathcal{F}(\Omega)\,. \tag{47}$$

Lemma

Assume

i. *the function* $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ *is* strictly decreasing, *and*

ii. $\exists$ *a constant* Ω_c $(|\Omega_c| < \infty)$ *such that* $\mathcal{F}(\Omega_c) = 0$,

then, at each interior point (in $\overline{\mathcal{I}}$*), where Eq.* (47), $\Phi = \mathcal{F}$, *is satisfied, the following holds*

$$\begin{aligned} A < \Omega \quad &\Longleftrightarrow \quad A < \Omega \leq \Omega_c \quad (A < \Omega_c) \\ A > \Omega \quad &\Longleftrightarrow \quad A > \Omega \geq \Omega_c \quad (A > \Omega_c) \\ A = \Omega \quad &\Longleftrightarrow \quad A = \Omega = \Omega_c \quad (A = \Omega_c)\,. \end{aligned}$$

(48)

Note 1: Due to (i), Ω_c [defined in (ii)] is unique. Also, observe, Ω_c exists and coincides with the (constant) value of Ω on the rotation axis, provided that part of the axis, $\varrho = 0$, is in the interior, $\mathcal{I}$, (i.e. if the rotating fluid does not have toroidal topology). This is because at points in $\{\varrho = 0\} \cap \mathcal{I} \neq \emptyset$, since $\Phi|_{\varrho=0} = 0$ and $\Phi = \mathcal{F}$ in $\mathcal{I}$, we have $\mathcal{F}(\Omega)|_{\varrho=0} = 0$; and $\mathcal{F}$ is, by requirement (i), invertible; yielding $\Omega|_{\varrho=0} = \text{const.} = \Omega_c$.

Note 2: Observe, if $\mathcal{F} \in C^1$ and $\mathcal{F}' < 0$, then, since $\partial_\Omega \Phi \geq 0$, Eq. (47) can be solved for Ω, by virtue of the implicit function theorem, yielding $\Omega = \Omega(\rho, U, B, A)$; and, by the regularity of the metric functions, (32), it follows in particular $\Omega \in C^1\left(\overline{\mathcal{I}}\right)$, requirement (11).

Note 3: It should be stressed that, since Φ is an *increasing* function in Ω, choosing the function $\mathcal{F}$ strictly *decreasing* [requirement (i)], Eq. (47) has a unique solution in Ω ("curve" solution with ϱ variable). Specially, this makes likely the existence of functions Ω, K, U, B, and A, solutions of the field equations and Eq. (14). Indeed, in numerical works concerning differential rotation the ansatz for the $\mathcal{F}$-law $\mathcal{F}(\Omega) = R_0^2(\Omega_c - \Omega)$, where R_0 is a free parameter describing the length of scale over which Ω changes, is generally used, and it is claimed that a solution exists. (See, e.g., Refs. 15-17.)

Proof. We consider a point $x \in \overline{\mathcal{I}}$ where the metric functions and the fluid angular velocity are solution, in particular, with reference to Eq. (47), the functions Φ and $\mathcal{F}$ valued at this point "intersect" each other, i.e.

$$\Phi(\varrho(x)\,,\,\Omega(x) - A(x)) = \mathcal{F}(\Omega(x)) \qquad (\forall x \in \overline{\mathcal{I}})\,.$$

From requirements (i) and (ii), it follows (at each interior point) $\mathrm{sign}(\mathcal{F}) = \mathrm{sign}(\Omega_c - \Omega)$. As regards Φ (at the interior point), on the axis ($\varrho = 0$) it obviously vanishes, cf. (46); following, from the relation $\Phi = \mathcal{F}$ (in $\overline{\mathcal{I}}$), $\Phi = \mathcal{F} = 0$ and, thus, $\Omega = \Omega_c$ on the axis. Outside the axis ($\varrho \neq 0$) we have $\mathrm{sign}(\Phi) = \mathrm{sign}(\Omega - A)$; and, hence, (outside the axis)

$$\begin{aligned}
\Phi = \mathcal{F} > 0 &\quad\Longleftrightarrow\quad A < \Omega < \Omega_c\,,\\
\Phi = \mathcal{F} < 0 &\quad\Longleftrightarrow\quad A > \Omega > \Omega_c\,,\\
\Phi = \mathcal{F} = 0 &\quad\Longleftrightarrow\quad A = \Omega = \Omega_c\,.
\end{aligned}$$

But also,

$$\begin{array}{lll}
\forall x \in \overline{\mathcal{I}} \quad / & A(x) < \Omega(x), & \begin{cases} A(x) < \Omega(x) < \Omega_c & \text{if } \varrho(x) \neq 0\ , \\ A(x) < \Omega(x) = \Omega_c & \text{if } \varrho(x) = 0\ , \end{cases} \\
\forall x \in \overline{\mathcal{I}} \quad / & A(x) > \Omega(x), & \begin{cases} A(x) > \Omega(x) > \Omega_c & \text{if } \varrho(x) \neq 0\ , \\ A(x) > \Omega(x) = \Omega_c & \text{if } \varrho(x) = 0\ , \end{cases} \\
\text{and } \forall x \in \overline{\mathcal{I}} \quad / & A(x) = \Omega(x), & A(x) = \Omega(x) = \Omega_c\,.
\end{array}$$

This yields (48). □

We are now in a position to get one of the main results of this work in the general differentially rotating case, namely, the following proposition.

Proposition 3

If the $\mathcal{F}$-law [in Eq. (14)] specifying the model of differential rotation is chosen such that

i. $\mathcal{F} : \mathbb{R} \longrightarrow \mathbb{R}$ *strictly decreasing,*

ii. $\exists$ *a constant* Ω_c $(|\Omega_c| < \infty)$ *such that* $\mathcal{F}(\Omega_c) = 0,$ *and*

iii. $\Omega_c > 0,$

then

$$0 < A < \Omega \leq \Omega_c \quad \textit{in } \overline{\mathcal{I}}\,; \tag{49}$$

in particular, $u^t u_\phi \geq 0$, *and, hence, the angular momentum density is non-negative. Moreover,*

$$0 < A < \max_{\mathcal{S}} \Omega \leq \Omega_c \quad \textit{in } \mathcal{E}\,. \tag{50}$$

Note: As remarked above, if the interior (fluid) contains points of the axis, then condition (ii) is already guaranteed, and Ω_c is the constant value of Ω on the axis; cf. Note 1 in the previous lemma. See also Notes 2 and 3. And observe, requirement (iii) is in principle much weaker than $\Omega > 0$, but, as seen in the conclusion of this proposition, $\Omega > 0$ already follows. Furthermore, the fact that $\Omega \leq \Omega_c$ in $\overline{\mathcal{I}}$ shows that in differentially rotating stars the core may rotate faster than the envelope, so that the core can be supported by rapid rotation before mass shedding is reached at the equator.[18]

Proof. We divide the proof in four steps.

First step: Let us see first $A \leq \Omega$ in $\overline{\mathcal{I}}$.

Suppose (to get a contradiction) $A(x_0) > \Omega(x_0)$ for some $x_0 \in \overline{\mathcal{I}}$. We have seen, in the previous lemma, cf. (48), that this is equivalent to $A(x_0) > \Omega(x_0) \geq \Omega_c$; and, hence, using hypothesis (iii), $A(x_0) > \Omega_c > 0$. Therefore, by the continuity of A [indeed $A \in C^1$, cf. (32)] and the asymptotic flatness $[\lim_{||x|| \to \infty} A = 0$, cf. (33)], we infer that

$\exists$ an open and connected neighborhood of x_0, $\mathcal{N}_{x_0} \subset \mathbb{R}^5$, such that

$$\begin{aligned} A &> \Omega_c \quad \text{in } \mathcal{N}_{x_0}\,, \\ \text{and} \quad A &= \Omega_c \quad \text{on } \partial\mathcal{N}_{x_0}\,. \end{aligned} \tag{51}$$

We distinguish two cases:

Case 1: $\mathcal{N}_{x_0} \cap \mathcal{E} = \emptyset$, that is, the neighborhood is contained in the interior, $\mathcal{N}_{x_0} \subseteq \mathcal{I}$.

Thus we have, again using the previous lemma, $A > \Omega \geq \Omega_c > 0$ in $\mathcal{N}_{x_0}$; particularly, $\Omega - A < 0$ in $\mathcal{N}_{x_0}$, and, therefore, by Eq. (29),

$$L_0\, A > 0 \qquad \text{in } \mathcal{N}_{x_0} \subseteq \mathcal{I}\,.$$

From (32), in particular, $A \in C^2(\mathcal{I}) \cap C^0(\overline{\mathcal{I}})$, and, hence, $A \in C^2(\mathcal{N}_{x_0}) \cap C^0(\overline{\mathcal{N}}_{x_0})$. And applying the weak maximum principle, Theorem 1 in Appendix A, to the operator L_0 (on A) in $\mathcal{N}_{x_0}$, we obtain that the maximum of A is reached on the boundary, i.e.

$$\max_{\overline{\mathcal{N}}_{x_0}} A = \max_{\partial\mathcal{N}_{x_0}} A\,,$$

contradicting (51).

Case 2: $\mathcal{N}_{x_0} \cap \mathcal{E} \neq \emptyset$. (Notice, here is included the case where $x_0 \in \partial\mathcal{I} = \mathcal{S}$.) We denote

$$\begin{aligned} \mathcal{I}_{x_0} &\equiv \mathcal{N}_{x_0} \cap \mathcal{I} \quad \subseteq \mathcal{I} \\ \mathcal{E}_{x_0} &\equiv \mathcal{N}_{x_0} \cap \mathcal{E} \quad \subset \mathcal{E} \\ \\ \text{and} \quad \Gamma &\equiv \mathcal{N}_{x_0} \cap \mathcal{S}\,. \end{aligned}$$

Observe, $\Gamma \neq \emptyset$, because $\mathcal{N}_{x_0}$ is connected, $x_0 \in \mathcal{N}_{x_0} \cap \overline{\mathcal{I}} \neq \emptyset$, and, by assumption in this case, $\mathcal{N}_{x_0} \cap \mathcal{E} \neq \emptyset$. Moreover, $\Gamma = \overline{\mathcal{I}}_{x_0} \cap \overline{\mathcal{E}}_{x_0} = \partial\mathcal{I}_{x_0} \cap \partial\mathcal{E}_{x_0}$.

Thus, we have

in the interior, from (51),

$$\begin{aligned} A &> \Omega_c \quad \text{in } \mathcal{I}_{x_0} \cup \Gamma \quad \subset \mathcal{N}_{x_0} \qquad (\Gamma \subset \partial\mathcal{I}_{x_0}) \\ A &= \Omega_c \quad \text{on } \partial\mathcal{I}_{x_0} \setminus \Gamma \quad \subset \partial\mathcal{N}_{x_0}\,; \end{aligned}$$

and, applying the weak maximum principle (Theorem 1 in Appendix A) to the differential inequality [cf. (29)]

$$L_0\, A > 0 \qquad \text{in } \mathcal{I}_{x_0} \subset \mathcal{I}$$

[again using (48), $A > \Omega \geq \Omega_c$ in $\mathcal{I}_{x_0}$], with $A \in C^2(\mathcal{I}_{x_0}) \cap C^0(\overline{\mathcal{I}}_{x_0})$, it follows

$$\max_{\overline{\mathcal{I}}_{x_0}} A = \max_{\partial\mathcal{I}_{x_0}} A = \max_{\Gamma} A\,;$$

in the exterior, we have analogously, from (51),

$$\begin{aligned} A &> \Omega_c \quad \text{ in } \mathcal{E}_{x_0} \cup \Gamma \quad \subset \mathcal{N}_{x_0} \qquad (\Gamma \subset \partial\mathcal{E}_{x_0}) \\ A &= \Omega_c \quad \text{ on } \partial\mathcal{E}_{x_0} \setminus \Gamma \quad \subset \partial\mathcal{N}_{x_0}\,. \end{aligned}$$

(Note, the point ∞ is not included in $\partial\mathcal{E}_{x_0}$, because $\Omega_c > 0$ and A is asymptotically flat, $A|_\infty = 0$.) But, in the exterior, $\mathcal{E}$, $\psi^2 \equiv 0$, and we have the elliptic equation for A

$$L_0\, A = 0 \qquad \text{in } \mathcal{E}_{x_0} \subset \mathcal{E}\,;$$

as a consequence, again by virtue of the maximum principle (Theorem 1 in Appendix A) now in $\mathcal{E}_{x_0}$

$$\max_{\overline{\mathcal{E}}_{x_0}} A = \max_{\partial\mathcal{E}_{x_0}} A = \max_{\Gamma} A\,.$$

We therefore have

$$\max_{\overline{\mathcal{I}}_{x_0}} A = \max_{\overline{\mathcal{E}}_{x_0}} A = \max_{\Gamma} A =: A(x_1), \quad \text{for some } x_1 \in \Gamma\,.$$

Thus, $\left(\overline{\mathcal{N}}_{x_0} = \overline{\mathcal{I}}_{x_0} \cup \overline{\mathcal{E}}_{x_0}\right)$

$$\max_{\overline{\mathcal{N}}_{x_0}} A = A(x_1), \quad \text{for some } x_1 \in \Gamma \subset \mathcal{N}_{x_0} \quad (x_1 \text{ interior point})\,;$$

and, since $A \in C^1(\mathbb{R}^5)$, in particular, $A \in C^1(\mathcal{N}_{x_0})$, it follows

$$\nabla A\big|_{x_1} = \mathbf{0}\,. \tag{52}$$

However, this is not possible, because, on the other hand, x_1 is a point of the star's surface $x_1 \in \Gamma \subset \mathcal{S}$, and, from the assumptions on the stellar model,

$\partial\mathcal{I} = \mathcal{S}$ satisfies an exterior sphere condition everywhere, i.e. $\partial\mathcal{E} = \mathcal{S} \cup \{\infty\}$ satisfies at each point of $\mathcal{S}$ (in particular, at x_1) an interior sphere condition (cf. Definition in Appendix A). This allows us to apply the so-called boundary-point lemma, Theorem 2 in Appendix A, for the operator L_0 in the exterior domain $\mathcal{E}_{x_0}$, with respect to the point $x_1 \in \Gamma \subset \partial\mathcal{E}_{x_0}$, being $A(x_1) = \max A$ in $\overline{\mathcal{E}}_{x_0}$. And, since $A \not\equiv$ const. [because $A > \Omega_c$ in $\mathcal{E}_{x_0} \cup \Gamma$, $A = \Omega_c$ on $\partial\mathcal{E}_{x_0} \setminus \Gamma$, and A is continuous], this yields

$$\langle \boldsymbol{\nu}, \nabla A\rangle\big|_{x_1} = \partial_{\boldsymbol{\nu}} A\big|_{x_1} \neq 0, \qquad \boldsymbol{\nu} \equiv \text{ outward pointing unit normal to } \mathcal{S} \text{ at } x_1\,;$$

contradicting (52). Consequently,

$$A \leq \Omega \quad \text{in } \overline{\mathcal{I}}\,. \tag{53}$$

Second step: $A < \Omega \leq \Omega_c$ in $\overline{\mathcal{I}}$.

This can be seen as follows. From inequality (53) and using (48) we also have

$$\Omega \leq \Omega_c \quad \text{in } \overline{\mathcal{I}}\,, \tag{54}$$

and, combining (53) and (54),

$$A \leq \Omega_c \quad \text{in } \overline{\mathcal{I}}\,. \tag{55}$$

On the other hand, we have Eq. (29), i.e. $L_0\, A = -\psi^2 \cdot (\Omega - A)$, satisfied everywhere, in particular, in the interior (in a classical sense). Let

$$u(x) := A(x) - \Omega_c \qquad \forall x \in \overline{\mathcal{I}}\,.$$

Since Ω_c is constant, we can rewrite Eq. (29),

$$Lu := L_0 u - \psi^2 \cdot u = +\psi^2 \cdot (\Omega_c - \Omega) \geq 0 \quad [\text{by } (54)]\,.$$

Hence, we have

$$Lu \geq 0 \quad \text{in } \mathcal{I}\,, \tag{56}$$

where, like A, $u \in C^2(\mathcal{I}) \cap C^0(\overline{\mathcal{I}})$, and

$$u \leq 0 \quad \text{in } \overline{\mathcal{I}}\,, \tag{57}$$

by inequality (55). We want to see $u < 0$. Suppose (to get a contradiction) that $u(\hat{x}) = 0$ for some $\hat{x} \in \mathcal{I}$; then, by (57), $0 = u(\hat{x}) = \max_{\overline{\mathcal{I}}} u$, $\hat{x} \in \mathcal{I}$

(interior point). However, by the strong maximum principle, Theorem 3 in Appendix A, for (56), u cannot reach a non-negative maximum at an interior point of $\mathcal{I}$, unless u is a constant in $\mathcal{I}$. That means, in our case, u cannot vanish somewhere in $\mathcal{I}$ unless it vanishes identically in $\mathcal{I}$. But $u \equiv \text{const.} = u(\hat{x}) = 0$ in $\mathcal{I}$, i.e.

$$A \equiv \text{ const.} = \Omega_c > 0 \quad \text{in } \mathcal{I}, \qquad A \in C^1 \text{ everywhere}\,, \tag{58}$$

yields, in particular,

$$\nabla A = \mathbf{0} \quad \text{on } \partial\mathcal{I} = \mathcal{S}\,. \tag{59}$$

On the other hand, in the exterior, $\mathcal{E}$, we have $L_0\, A = 0$, with $A \in C^2(\mathcal{E}) \cap C^0(\overline{\mathcal{E}})$, and, by the weak maximum principle (Theorem 1 in Appendix A)

$$\max_{\overline{\mathcal{E}}} A \;=\; \max_{\partial\mathcal{E}=\mathcal{S}\cup\{\infty\}} A\,,$$

but, using asymptotic flatness ($A|_\infty = 0$) and (58), actually,

$$\max_{\overline{\mathcal{E}}} A = \max_{\mathcal{S}} A =: A(x_1) \quad \text{for some } x_1 \in \mathcal{S} \subset \partial\mathcal{E}\,.$$

In particular, since $A \not\equiv$ const., the boundary-point lemma, Theorem 2 in Appendix A, applied to the operator L_0 in the exterior domain $\mathcal{E}$ (where, by assumption, an interior sphere condition is satisfied in particular at $x_1 \in \mathcal{S} \subset \partial\mathcal{E}$) yields a non-vanishing outward normal derivative

$$\langle \boldsymbol{\nu}, \nabla A\rangle|_{x_1} = \partial_{\boldsymbol{\nu}} A|_{x_1} \neq 0\,,$$

in contradiction to (59). Therefore, $u < 0$ everywhere in $\mathcal{I}$, i.e. $A < \Omega_c$ in $\mathcal{I}$; and, hence, also on $\partial\mathcal{I}$, because, by the weak minimum principle, Theorem 1 in Appendix A, applied to $L_0\, A = -\psi^2 \cdot (\Omega - A) \leq 0$ in $\mathcal{I}$ [by (53)], we get $\min_{\overline{\mathcal{I}}} A = \min_{\partial\mathcal{I}} A$. Therefore, $A < \Omega_c$ in $\overline{\mathcal{I}}$, or, equivalently [cf. (48)],

$$A < \Omega \leq \Omega_c \quad \text{in } \overline{\mathcal{I}}\,.$$

Third step: $A > 0$ everywhere (i.e. the same conclusion of Proposition 1, but now using different hypotheses).

We have seen in the first step $A \leq \Omega$ in $\overline{\mathcal{I}}$; which yields, $L_0\, A \leq 0$ in $\overline{\mathcal{I}}$. On the other hand, $L_0\, A = 0$ in $\mathcal{E}$. Accordingly,

$$L_0\, A \leq 0 \quad \text{everywhere in } \overline{\mathcal{I}} \cup \mathcal{E} = \mathbb{R}^5\,.$$

Applying now the strong minimum principle for generalized supersolutions, Theorem 4 in Appendix B, and using asymptotic flatness, as was argued in the proof of Proposition 1, it follows $A > 0$ everywhere. [Notice, here $A \not\equiv$ const., because, by asymptotic flatness and continuity, $A \equiv$ const. is equivalent to $A \equiv 0$; by Eq. (37), also $\Omega \equiv 0$, and, hence, from $\Omega - A \equiv 0$, we would have [cf. (46)] $0 \equiv \Phi = \mathcal{F}(0)$; but this is not possible, since requirements (i) and (ii) imply $\mathcal{F}(0) > \mathcal{F}(\Omega_c) = 0$.]

Thus, $A > 0$ everywhere, in particular, in the interior; using now the result of the second step, we finally get (49), $0 < A < \Omega \leq \Omega_c$ in $\overline{\mathcal{I}}$. Notice, hence, $\Omega > 0$ (in $\overline{\mathcal{I}}$).

Fourth step: $A < \max_{\mathcal{S}} \Omega \leq \Omega_c$ in $\mathcal{E}$.

The elliptic equation holding in the exterior,

$$L_0 A = 0 \quad \text{in } \mathcal{E}\,,$$

yields, by virtue of the weak maximum principle (Theorem 1 in Appendix A),

$$\max_{\overline{\mathcal{E}}} A \;=\; \max_{\partial\mathcal{E}=\mathcal{S}\cup\{\infty\}} A\,;$$

but, using asymptotic flatness ($A|_{\infty} = 0$), we actually have $\max_{\mathcal{S}\cup\{\infty\}} A = \max_{\mathcal{S}} A$. On the other hand, we have seen $A < \Omega \leq \Omega_c$ in particular in $\partial\mathcal{I} = \mathcal{S}$, and $\mathcal{S}$ is compact. Hence,

$$\text{in } \mathcal{E}, \quad 0 < A \leq \max_{\overline{\mathcal{E}}} A = \max_{\mathcal{S}=\partial\mathcal{I}} A < \max_{\mathcal{S}} \Omega \leq \Omega_c\,,$$

establishing also (50). □

Remark 3. Analogously as argued in Remarks 1 and 2, it is possible to "reflect" Proposition 3. As a consequence, in particular, in a model for a star which is rotating differentially with the function $\mathcal{F}$ either strictly decreasing with $\Omega_c > 0$ or strictly increasing with $\Omega_c < 0$, $\mathcal{F}(\Omega_c) = 0$, the angular momentum density has the same sign as Ω_c, and, hence, as the angular velocity of the fluid. Also, accordingly, the following holds

$$0 < |A| < |\Omega| \leq |\Omega_c| \quad \text{in } \overline{\mathcal{I}}\,,$$

$$0 < |A| < \left|\max_{\mathcal{S}} \Omega\right| \leq |\Omega_c| \quad \text{in } \mathcal{E}\,.$$

A. Preliminary observation

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, $V : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and $G \subset \mathbb{R}^n$ a domain where Gauss' theorem can be applied. Then, due to

$$\mathrm{div}(uV) = \sum_i \partial_i(uV_i) = \langle \mathcal{D}u, V\rangle + u\,\mathrm{div}V$$

[where $\langle .,.\rangle$ is the Euclidean scalar product, $\mathcal{D}$ is the gradient operator, and $\mathrm{div}V := \sum_i \partial_i(V_i)$, the divergence], we get, integrating over G and applying the Gauss theorem,

$$\int_G u\,\mathrm{div}V = -\int_G \langle \mathcal{D}u, V\rangle + \int_{\partial G} u\,\langle V, \boldsymbol{\nu}\rangle\,, \tag{60}$$

where $\boldsymbol{\nu}$ is the outer unit normal of ∂G (and, for simplicity in the notation, volume- and surface elements have been dropped).

B. Appropriate form of the field equation

The general (elliptic) field equation for A, Eq. (19), may be rewritten as follows:

$$\mathrm{div}(\rho^3 e^{3B} e^{-4U}\mathcal{D}A) = -f^2\cdot(\Omega - A)\,, \tag{61}$$

where $\mathcal{D} := (\partial_\rho, \partial_z)$ and 'div' are the flat expressions in $\mathbb{R}^2$, and

$$f^2(\rho,z) \equiv f^2 := \rho^3 e^{3B-4U}\psi^2 = 16\pi\,(\varepsilon + p)\,\frac{\rho^3 e^{3B} e^{2(K-2U)}}{e^{2U} - \rho^2 e^{2(B-U)}(\Omega - A)^2} \geq 0\,.$$

Especially we have $f^2 \equiv 0$ in the exterior E of the star [cf. (5)].

Note, in this section (independent of Sec. IV) we go back to the field equation in the meridian plane coordinates, $(\rho, z) \in \mathbb{R}_0^+ \times \mathbb{R}$, instead of the 5-*lifted* one (on $\mathbb{R}^5$) (cf. Sec. III A).

C. Main observation

Multiplying Eq. (61) by A, and using Eq. (60), by setting $u = A$, $V = \rho^3 e^{3B} e^{-4U}\mathcal{D}A$, and $G = \mathbb{R}_0^+ \times \mathbb{R} \subset \mathbb{R}^2$ [actually, we consider a ball in $\mathbb{R}^2$ centered at the origin of the coordinate system with arbitrarily large radius, $\mathcal{B}_\sigma(\mathbf{0}) \subset \mathbb{R}^2$, and take $G = \mathcal{B}_\sigma(\mathbf{0}) \cap (\mathbb{R}_0^+ \times \mathbb{R}) \subset \mathbb{R}^2$, $\sigma \to \infty$], we obtain

$$-\int_I f^2 A(\Omega - A) = -\int_{\mathbb{R}_0^+\times\mathbb{R}} \rho^3 e^{3B} e^{-4U}\|\mathcal{D}A\|^2 + \int_{\partial(\mathbb{R}_0^+\times\mathbb{R})} A\,\rho^3 e^{3B} e^{-4U}\langle \mathcal{D}A, \nu\rangle\,,$$

where $I \subset \mathbb{R}_0^+ \times \mathbb{R} \subset \mathbb{R}^2$ represents the (ρ, z)-coordinates of the interior of the star (note, f^2 vanishes in the exterior E). The first term on the right hand side (which converges, since $\|\mathcal{D}A\|$ falls off rapidly enough at the space-like "infinity"[19]) is obviously non-positive. And the second term vanishes, because of the asymptotic behavior at spatial infinity of the metric functions, specially of A,[19] and because the integrand, due to the factor ρ^3, vanishes on the axis of rotation $\rho = 0$, which is the other part of $\partial(\mathbb{R}_0^+ \times \mathbb{R}) = \{R = (\rho^2 + z^2)^{1/2} \to \infty\} \cup \{\rho = 0\}$. Hence, we have found

$$\int_I f^2 A\,(\Omega - A) \geq 0\,. \tag{62}$$

D. Consequences

In order to see more the linear algebra behind, we introduce now the bilinear form

$$\langle u, v\rangle_f := \int_I f^2(\rho, z)\, u(\rho, z)\, v(\rho, z)\, d\rho\, dz\,, \qquad u, v : I \to \mathbb{R} \quad \text{in } L^2(I)\,,$$

and the induced semi-norm $\|.\|_f := \left(\langle .\,, . \rangle_f\right)^{1/2}$. With this definition we can write inequality (62) as

$$\langle A, \Omega\rangle_f \geq \|A\|_f^2\,,$$

and immediately see that especially

$$\langle \Omega, A\rangle_f = \langle A, \Omega\rangle_f \geq 0\,. \tag{63}$$

Furthermore, using the Cauchy-Schwarz inequality, we have $\|A\|_f\|\Omega\|_f \geq \langle A, \Omega\rangle_f \geq \|A\|_f^2$, and hence (since $A \equiv 0 \iff \Omega \equiv 0$) we get (in full general) the main result of these sections, namely,

$$0 \leq \|A\|_f \leq \|\Omega\|_f\,, \tag{64}$$

i.e.

Proposition 4

$$\boxed{0 \leq \int_I f^2 A^2 \leq \int_I f^2\, \Omega^2} \tag{65}$$

[without any restriction concerning the rotation law, $\Omega \mapsto \mathcal{F}(\Omega)$ in (14), in the differentially rotating case, nor the regularity and sign uniformity of Ω]. These integrals can be regarded as some kind of "mean value" with respect to the "density" f^2, thus, (65) fulfilling the physical expectations.[2]

In addition, multiplying inequality (64) by $\|\Omega\|_f$, we find (again using the Cauchy-Schwarz inequality) $\|\Omega\|_f^2 \geq \|\Omega\|_f \|A\|_f \geq \langle\Omega, A\rangle_f$, i.e.

$$\langle\Omega, \Omega - A\rangle_f \geq 0\,. \tag{66}$$

Remarkably, the integral given in (66) has an important physical meaning; it is, up to a constant factor, the so-called *total rotational energy* (see, e.g., Refs. 15, 16, 18, and 20),

$$T \equiv \frac{1}{2}\int_I \Omega\, dJ \;=\; \frac{1}{2}\int_I 2\pi\,\Omega\, T_\phi^{\ t}\,(-\mathrm{g})^{1/2} d\rho\, dz \;=\; \frac{1}{16}\langle\Omega, \Omega - A\rangle_f$$

(also denoted E_{rot} or M_{rot}). Thus, (66) shows $T \geq 0$. Furthermore,

$$16\,T = \langle\Omega, \Omega - A\rangle_f = \|\Omega\|_f^2 - \langle\Omega, A\rangle_f \;\leq\; \|\Omega\|_f^2,$$

by (63). Hence, we have the following proposition.

Proposition 5

$$\boxed{0 \;\leq\; T \equiv \frac{1}{16}\int_I f^2\,\Omega\,(\Omega - A) \;\leq\; \frac{1}{16}\int_I f^2\,\Omega^2 \quad .} \tag{67}$$

This generalizes the result given by Hartle (cf. Ref. 7, Sec. IV) in the limit of slow (differential) rotation to the general differentially rotating case. Specifically, an alternative and much simpler proof of the referenced result (in the slow rotation limit) is possible in a way (cf. Ref. 21) which can be even generalized (outside this limit).

VI. CONCLUSIONS

Aiming to derive general properties of equilibrium non-singular stellar models with differential rotation, we have established that for a wide class of rotation laws the distribution of angular velocity of the fluid has a sign, and then both the dragging rate (angular velocity of locally non-rotating observers) and the angular momentum density have the sign of the fluid angular velocity (Sec. IV). In addition, the mean value (with respect to a density function) of the dragging rate is shown to be less than the mean value of the fluid angular velocity; and this is proved in full general, without having to restrict the rotation law, nor the uniformity in sign of the fluid angular velocity. A further simple calculation of linear algebra on this inequality yields a generalization of the result given by Hartle[7] concerning positivity and upper bound of the total rotational energy in the limit of slow (differential) rotation to the general differentially rotating case (Sec. V).

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APPENDIX A: Maximum (minimum) principles for classical sub-(super) solutions

By G we denote an open and connected set, i.e. a domain, in $\mathbb{R}^n$, $n \geq 2$. The boundary is denoted by $\partial G \equiv \overline{G} \cap (\mathbb{R}^n \setminus G)$. We define the differential operators

$$L_0 u \;:=\; a_{ij}(x)\, \partial_{ij} u \,+\, b_i(x)\, \partial_i u\,, \qquad a_{ij} = a_{ji}\,,$$

$$\text{and} \quad Lu \;:=\; L_0 u \,+\, c(x) \cdot u$$

(where the summation convention that repeated indices indicate summation from 1 to n is followed), such that[22]

1. L (and, hence, L_0) is uniformly elliptic in G in the special form

$$0 < \lambda\, |y|^2 \leq a_{ij}(x)\, y_i y_j \leq \Lambda\, |y|^2, \quad \forall y \in \mathbb{R}^n \backslash \{0\},\ \forall x \in G \qquad \left(|y|^2 := \sum_i {y_i}^2 \right), \tag{A1}$$

   where λ and Λ are constants such that $0 < \lambda \leq \Lambda < \infty$;

2. all coefficients in L (and in L_0), a_{ij}, b_i (for all i and j), and c, are measurable and bounded functions in G,

$$|a_{ij}| < \infty, \quad |b_i| < \infty, \quad |c| < \infty \ \text{ in } G \quad (i, j \in \{1, \ldots, n\})\,. \tag{A2}$$

Theorem 1 (the weak maximum (minimum) principle for L_0 $(c = 0)$)
Suppose that $L_0 u \geq 0$ (≤ 0) in a bounded domain G, with $u \in C^2(G) \cap C^0(\overline{G})$. Then the maximum (minimum) of u is attained on the boundary, that is,

$$\max_{\overline{G}} u = \max_{\partial G} u \quad \left(\min_{\overline{G}} u = \min_{\partial G} u \right) .$$

(A proof of that theorem can be found, e.g., in Ref. 23, Theorem 3.1.)

Definition
For a set $G \subset \mathbb{R}^n$, the boundary ∂G is said to satisfy an **interior (exterior) sphere condition** *at a point $x_1 \in \partial G$ iff there exists a ball $B \subset G$ $\;(B \subset \mathbb{R}^n \setminus \overline{G})$ with $x_1 \in \partial B$*

Theorem 2 (the boundary-point lemma)
Suppose that $L_0 u \geq 0$ $(c = 0)$ in a domain G not necessarily bounded.
Let $x_1 \in \partial G$ be such that
(i) *u is continuous at x_1,*
(ii) *$u(x_1) \geq u(x)$ for all $x \in G$, and*
(iii) *∂G satisfies an* interior sphere condition *at x_1.*
Then the outer normal derivative of u at x_1, if it exists, satisfies the strict inequality

$$\partial_{\boldsymbol{\nu}} u(x_1) > 0,$$

unless $u \equiv \text{const.} = u(x_1)$.

(A proof of that result can be found, e.g., in Ref. 24, Theorem 7, Chap. 2.)

If $c \leq 0$ (in $Lu \geq 0$), the same conclusion holds provided $u(x_1) \geq 0$.

(See Ref. 24, Theorem 8, Chap. 2. Also Ref. 23, Lemma 3.4.)

Theorem 3 (the strong maximum (minimum) principle for L)
Let $Lu \geq 0$ (≤ 0) in a domain G not necessarily bounded, with $u \in C^2(G) \cap C^0(\overline{G})$, and the operator L satisfying

$$c \leq 0 \quad \text{in } G \tag{A3}$$

apart from conditions (A1) and (A2) above.
Then u cannot attain a non-negative maximum (non-positive minimum) at an interior point of G, unless $u \equiv \text{const.}$ in G.

For $c = 0$, i.e. $L = L_0$, the same conclusion holds without the requirement 'non-negative' ('non-positive').

(For the proof we refer again to Ref. 23, Theorem 3.5; or Ref. 24, Theorems 5 and 6, Chap. 2.)

APPENDIX B: Maximum (minimum) principle for generalized sub-(super) solutions

Consider in a domain (open and connected set) $G \subset \mathbb{R}^n \quad (n \geq 2)$ the differential operator with principal part of divergence form, defined by

$$Lu = \partial_i[a_{ij}(x)\partial_j u + a_i(x)\, u] + b_i(x)\, \partial_i u + c(x)\, u\,,$$

with $a_{ij} = a_{ji}$. Notice, an operator L of the general form $Lu = \tilde{a}_{ij}(x)\partial_{ij}u + \tilde{b}_i(x)\partial_i u + \tilde{c}(x)u$ may be written in divergence form provided its principal coefficients $\tilde{a}_{ij}$ are differentiable. If furthermore the $\tilde{a}_{ij}$ are constants, then even with coinciding coefficients ($a_{ij} = \tilde{a}_{ij}$, $b_i = \tilde{b}_i$, $c = \tilde{c}$) and $a_i \equiv 0$. Let us assume that

1. L is strictly elliptic in G, i.e. $\exists$ a constant $\lambda > 0$ such that $\lambda \leq$ the minimum eigenvalue of the principal coefficient matrix $[a_{ij}(x)]$,

$$\lambda\, |y|^2 \leq a_{ij}(x)\, y_i y_j \qquad \forall y \in \mathbb{R}^n, \quad \forall x \in G\,; \tag{B1}$$

2. a_{ij}, a_i, b_i, and c are measurable and bounded functions in G,

$$|a_{ij}| < \infty, \quad |a_i| < \infty, \quad |b_i| < \infty, \quad |c| < \infty \ \text{ in } G \quad (i, j \in \{1, \ldots, n\})\,. \tag{B2}$$

By definition, for a function u which is only assumed to be *weakly differentiable* and such that the functions $a_{ij}\partial_j u + a_i u$ and $b_i \partial_i u + cu$, $i = 1, \ldots, n$ are locally integrable [in particular, for u belonging to the Sobolev space $W^{1,2}(G)$], u is said to satisfy $Lu = g$ in G *in a generalized (or weak) sense* (g also a locally integrable function in G) if it satisfies

$$\begin{aligned}\mathcal{L}(u, \varphi; G) \ &:= \ \int_G \{(a_{ij}\partial_j u + a_i u)\partial_i \varphi - (b_i \partial_i u + cu)\varphi\} dx \\ &= \ -\int_G g\, \varphi\, dx, \qquad \forall \varphi \geq 0 \quad \varphi \in C_c^1(G)\end{aligned}$$

[where $C^1_c(G)$ is the set of functions in $C^1(G)$ with compact support in G].

Notice, u is *generalized sub-(super-)solution* relative to a differential operator L and the domain G [i.e. satisfies $Lu \geq 0$ (≤ 0) in G in a generalized sense] if it satisfies $\mathcal{L}(u, \varphi; G) \leq 0$ (≥ 0), $\quad \forall\varphi \geq 0 \;\; \varphi \in C^1_c(G)$.

Theorem 4 (strong maximum (minimum) principle)
Let $u \in W^{1,2}(G) \cap C^0(G)$ satisfy $Lu \geq 0$ (≤ 0) in G in a generalized sense, with

$$\int_G (c\varphi - a_i \partial_i \varphi)\, dx \leq 0, \quad \forall\varphi \geq 0 \quad \varphi \in C^1_c(G)\,. \tag{B3}$$

[equivalent to requirement (A3) *in the classical case] and conditions* (B1) *and* (B2) *above.*
Then u cannot achieve a non-negative maximum (non-positive minimum) in the interior of G, unless $u \equiv$ const.

(A proof of this theorem can be found in Ref. 23, Theorem 8.19.)

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Bounds on the dragging rate of slowly and differentially rotating relativistic stars

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Abstract

For relativistic stars rotating slowly and differentially with a positive angular velocity, some properties in relation to the positiveness of the rate of rotational dragging and of the angular momentum density are derived. Moreover, the proof for the bounds on the rotational mass-energy, which we have generalized (outside the slow rotation limit) in a previous paper, is briefly exposed.

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I. INTRODUCTION

In a previously published paper[1] we have given general bounds on the dragging rate (angular velocity of locally non-rotating observers, or angular velocity of cumulative dragging) of a differentially rotating relativistic stellar configuration; however, the validity of these bounds depends heavily on the underlying *rotation law*, which must be compatible with the field equations.

In the prescription for calculating a *slowly* and differentially rotating relativistic stellar configuration the field equations are expanded in powers of a *fluid angular velocity parameter*, and the perturbations (around a non-rotating configuration) are calculated by retaining only first and second order terms. Hartle[2] has derived these equations of structure in the rigidly rotating case. It is remarkable that at first order the only effect of the rotation is to drag the inertial frames; at second order it also deforms the star. An expansion of the dragging rate potential in powers of the angular velocity parameter only contains odd powers. Hence, if one is interested in calculating all effects up to second order, it is then sufficient to include only the linear (first order) corrections in the dragging rate. And it turns out that up to first order there are no restrictions on the rotation profile by the field equations, more exactly, by the Euler equation, through a rotation law. Hence, the result on bounds on the dragging rate in the general (differentially) rotating case, mentioned above (Proposition 3 in Ref. 1), does not apply any more in the slow rotation limit. In the present paper the bounds on the dragging rate (including positivity of the angular momentum density) are refined for the slowly and differentially rotating case.

The first order equations of structure reduce to the time-angle field equation component (to first order), which is a partial differential equation, linear in the dragging rate potential. This linearity in the dragging rate persuades us to re-write that equation in appropriate coordinates —in order to avoid the coordinate singularity occurring on the axis of rotation in spherical polar coordinates, generally used in the slow rotation approximation— so that in the new coordinates the equation writes in a "regular" form as an elliptic equation with measurable and bounded coefficients. This allows us to apply a minimum principle for generalized supersolutions in the whole domain (interior and exterior of the star). Making use of the asymptotic flatness condition, this will lead us directly to the positivity of the dragging rate, provided that the distribution of angular velocity of the fluid is non-negative everywhere (and non-trivial) and that we start from a reasonable unperturbed (non-rotating) stellar model satisfying the weak energy condition. In this case, the linearity (in the rotation) of the considered equation will guarantee a positive angular momentum density, provided that the amplitude of the rotation profile is bounded in a certain way.

The *rotational mass-energy*, derived by Hartle in Ref. 3, although accurate to second order in the angular velocity, involves only quantities which can be calculated from the first order structure equation (time-angle component of the Einstein field equations) as well. A proof of the positivity and an upper bound on this rotational energy was given in the same paper,[3] however using an expansion in eigenfunctions (and leaving open the non-trivial mathematical problems which may arise on the existence of these eigenfunctions). We found a much simpler proof of that result, avoiding expansions in eigenfunctions (in the present paper briefly exposed) which, remarkably, was possible to generalize to the general differentially rotating case, i.e. outside the slow rotation limit (cf. Sec. V of Ref. 1).

The paper is organized as follows. After a description of the relativistic rotating stellar model in Sec. II, and a brief revision of the concepts of angular momentum density and rate of rotational dragging in Sec. III, in Sec. IV we concentrate on the slow rotation approximation, particularly on the first order perturbations of the metric (linear correction of the dragging rate, with description of the unperturbed, i.e. zero order, configuration), and explicit expressions for the expansions of the angular momentum density and of the rotational mass-energy are derived. In the same section the null contribution (at first order in the angular velocity) of the integrability condition of the Euler equation is discussed, and the time-angle component of the Einstein equations (to first order) is written in appropriate coordinates, as a background allowing to apply a minimum principle and obtain the first of the properties mentioned above and proved in Sec. V, and consequences of that one. Apart from this, we sketch here the alternative proof of the bounds on the rotational energy. Finally, in Sec. VI, concluding remarks are briefly stated.

II. THE RELATIVISTIC ROTATING STELLAR MODEL

The spacetime of a rotating relativistic star is represented by a Lorentzian 4-manifold $(\mathcal{M}, \boldsymbol{g})$ which satisfies the following

A. Assumptions

i. the spacetime is stationary in time and axially symmetric, which means that $\boldsymbol{g}$ admits two global Killing vector fields, a time-like future-directed one, $\boldsymbol{\xi}$, and a space-like one, with closed trajectories, $\boldsymbol{\eta}$, except on a time-like 2-surface (defining the axis of rotation) where $\boldsymbol{\eta}$ vanishes;

ii. the spacetime is asymptotically flat; in particular, $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) \to -1$, $\boldsymbol{g}(\boldsymbol{\eta}, \boldsymbol{\eta}) \to +\infty$, and $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\eta}) \to 0$ at spatial infinity [the signature of the metric $\boldsymbol{g}$ being $(- + ++)$];

iii. the matter —confined in a compact region in the space (interior), with vacuum on the outside, so that (ii) holds— is perfect fluid, and therefore the energy-momentum tensor (source of the Einstein equations) is written as

$$\mathbf{T} = (\varepsilon + p)\mathbf{u}^\flat \otimes \mathbf{u}^\flat + p\,\boldsymbol{g}\,,$$

where ε and p denote the energy density and the pressure of the fluid, respectively; and $\mathbf{u}^\flat$ denotes the 1-form equivalent to the 4-velocity of the fluid $\mathbf{u}$ (in the exterior $\mathbf{T} \equiv 0$; hence, $\varepsilon + p = p = 0$ there);

iv. the fluid velocity is azimuthal (non-convective) (*circularity condition*), i.e.

$$\mathbf{u}^\flat \wedge \boldsymbol{\xi}^\flat \wedge \boldsymbol{\eta}^\flat = 0\,;$$

v. $(\mathcal{M}, \boldsymbol{g})$ satisfies Einstein's field equations $\boldsymbol{g} = 8\pi\mathbf{T}$ for the energy-momentum tensor $\mathbf{T}$ of a perfect fluid (iii), where $\boldsymbol{g} \equiv \mathrm{R}\mathbf{ic} - 1/2\, R\,\boldsymbol{g}$ denotes the Einstein tensor —equations which can also be written in the form

$$\mathrm{R}\mathbf{ic} = 8\pi(\mathbf{T} - \frac{1}{2}\mathrm{t}r(\mathbf{T})\,\boldsymbol{g})\,; \tag{1}$$

vi. ε and p satisfy a barotropic (one-parameter) equation of state, $\varepsilon = \varepsilon(p)$;

vii. $\varepsilon + p \geq 0$ (*weak energy condition* for perfect fluid, assuming $\varepsilon \geq 0$);[4]

viii. the metric functions are essentially bounded.

B. Form of the metric

Assumptions (i) and (ii) imply that the two Killing fields commute, $[\boldsymbol{\xi}, \boldsymbol{\eta}] = 0$,[5] which is equivalent to the existence of coordinates $x^0 \equiv t$ and $x^1 \equiv \phi$ such that $\boldsymbol{\xi} \equiv \partial_t$ and $\boldsymbol{\eta} \equiv \partial_\phi$; moreover, by the circularity condition (iv), the spacetime $\boldsymbol{g}$ admits 2-surfaces orthogonal to the group orbits of the Killing fields (orthogonal transitivity).[6] We may then choose the two remaining coordinates (x^2, x^3) in one of these 2-surfaces and carry them to the whole spacetime along the integral curves of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$; accordingly, the metric can be written in the form

$$\begin{aligned} ds^2 &\equiv g_{\alpha\beta}dx^\alpha dx^\beta \\ &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{22}(dx^2)^2 + 2g_{23}dx^2dx^3 + g_{33}(dx^3)^2\,, \end{aligned}$$

where the metric coefficients are independent of the time $x^0 \equiv t$ and azimuthal $x^1 \equiv \phi$ coordinates corresponding to the two Killing fields; that is, $g_{\alpha\beta} = g_{\alpha\beta}(x^2, x^3)$.

When solving Einstein's field equations it is convenient to specify the coordinates x^2 and x^3 in such a way as to simplify the task; a particular choice, usually made when studying slowly rotating configurations,[2] is the one which makes $g_{23} = 0$ and $g_{33} = g_{\phi\phi} \sin^{-2} x^3$. Hence x^2 and x^3 are chosen so that at large spatial distances the asymptotically flat metric is expressed in terms of spherical polar coordinates in the usual way. In the resulting coordinate system, with the notation $x^2 \equiv r,\ x^3 \equiv \theta$, and with new symbols for the metric functions, the line element reads

$$ds^2 = -H^2 dt^2 + Q^2 dr^2 + r^2 K^2 [d\theta^2 + \sin^2\theta \, (d\phi - A dt)^2] \,, \tag{2}$$

where H, Q, K and A are functions of r and θ alone. In these coordinates ($r \geq 0, \quad 0 \leq \theta \leq \pi$) the spatial infinity is given by $r \to \infty$, and the axis of rotation ($\partial_\phi \equiv \boldsymbol{\eta} = 0$) is described by $\theta \to 0$ or $\pi \quad (r \geq 0)$.

Notice that the function A appears in the metric (2) as the non-vanishing of the ($t\phi$) metric component of a rotating configuration. A is actually the *dragging rate* potential [cf. Sec. III].

C. Differential rotation

According to assumption (iv) of Sec. II A —circularity condition—, the fluid 4-velocity $\mathbf{u}$ has the form

$$\mathbf{u} = u^t \partial_t + u^\phi \partial_\phi = u^t (\partial_t + \Omega \partial_\phi), \qquad \text{where} \ \ \Omega \equiv \frac{u^\phi}{u^t} = \frac{d\phi}{dt}$$

is the angular velocity of the fluid measured in units of t, i.e. as seen by an inertial observer at infinity whose proper time is the same as the coordinate t (observer ∂_t), and u^t is the normalization factor, such that $\boldsymbol{g}(\mathbf{u}, \mathbf{u}) = -1$, i.e. $(u^t)^{-2} = -(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})$. The star's matter rotates then in the azimuthal direction ϕ.

We consider a star rotating differentially, with a prescribed distribution of angular velocity

$$\Omega \equiv \Omega(x^2, x^3) \equiv \Omega(r, \theta) \,,$$

an essentially bounded function. However, with the assumptions made (Sec. II A), the rotation profile of the fluid cannot be freely chosen, this shows up in the following. We consider the equation of hydrostatic equilibrium, ${T^{\alpha\beta}}_{;\beta} = 0$ (integrability conditions of the field equations), particularly, its part orthogonal to the fluid 4-velocity $\mathbf{u}$, i.e. the Euler equation,

$$dp = -(\varepsilon + p)\,\mathbf{a}\,, \tag{3}$$

where $\mathbf{a}$ is the 4-acceleration of the fluid, $\mathbf{a}^\sharp = \nabla_{\mathbf{u}}\mathbf{u}$, specifically,

$$\mathbf{a} = -d\ln u^t + u^t u_\phi\, d\Omega\,. \tag{4}$$

And the integrability condition of Eq. (3), taking into account (vi) of Sec. II A, $\varepsilon = \varepsilon(p)$, is $d\mathbf{a} = 0$; following, from (4), $d(u^t u_\phi) \wedge d\Omega = 0$; in other words, the fluid angular velocity, Ω, is functionally related to the specific angular momentum times u^t,

$$u^t u_\phi = \mathcal{F}(\Omega)\,. \tag{5}$$

Nevertheless, as will be seen in Sec. IV B, in the slow rotation approximation, at first order in the angular velocity, Eq. (5) is no restriction on the rotation profile $\Omega(r,\theta)$.

III. ANGULAR MOMENTUM DENSITY AND DRAGGING RATE

The total angular momentum of a rotating relativistic star can be defined[7] from the variational principle for general relativity —for an isolated system which is not radiating gravitational waves—, but this is shown[7] to coincide with the *geometrical* definition —from the asymptotic form of the metric at large space-like distances from the rotating fluid— (analog to the ADM mass), which for stationary and axisymmetric (asymptotically flat) spacetimes is given by the Komar integral for the angular momentum,[8] a surface integral, which, when reformulated using the Gauss theorem and the Einstein equations, converts into the volume integral over the interior

$$J = \int_{\mathcal{I}} T_\alpha{}^\beta \eta^\alpha n_\beta dv \tag{6}$$

$$= \int_{\mathcal{I}} T_\phi{}^t n_t dv \;= \int_{\mathcal{I}} T_\phi{}^t (-\mathrm{g})^{1/2} d^3x\,, \tag{7}$$

where $\mathbf{T}$ is the energy-momentum tensor of perfect fluid, $\boldsymbol{\eta}$ is the Killing field corresponding to the axial symmetry, $\mathbf{n}^\sharp$ is the unit time-like and future pointing normal to the hypersurface of constant t, i.e. $\mathbf{n} = n_t\, dt$, with $n_t > 0$, and dv is the proper volume element of the surface $t =$ const., i.e. $\int_{\mathcal{I}} dv =$ Vol, the volume of the body of the star, $\mathcal{I} \equiv$ interior of the star ($t =$ const.). Here $\mathrm{g} \equiv \det(\boldsymbol{g})$. The invariantly defined integrand of this volume integral (6), $T_\alpha{}^\beta \eta^\alpha n_\beta$, is what one would naturally define as *angular momentum density* —coinciding with the standard form in special relativity—, and can be calculated

$$
\begin{aligned}
T_\alpha{}^\beta \eta^\alpha \, n_\beta &= n_t \, T_\phi{}^t \\
&= n_t \, (\varepsilon + p) u^t u_\phi \qquad [u_\phi = u^t (g_{t\phi} + \Omega g_{\phi\phi})] \\
&= n_t \, (\varepsilon + p) (u^t)^2 (g_{t\phi} + \Omega g_{\phi\phi}) \\
&= n_t \, (\varepsilon + p) (u^t)^2 g_{\phi\phi} (\Omega - A), \qquad \text{with} \;\; n_t = H = \left(\frac{-g_{tt} g_{\phi\phi} + g_{t\phi}^2}{g_{\phi\phi}} \right)^{1/2},
\end{aligned}
\tag{8}
$$

where A is the metric function [cf. (2)] such that

$$
g_{t\phi} = -A \; g_{\phi\phi} \, . \tag{9}
$$

It is remarkable that, since $n_t > 0$, $g_{\phi\phi} \geq 0$, and we are assuming the energy condition $\varepsilon + p \geq 0$ [(vii) in Sec. II A], the sign of the angular momentum density (8) is determined by the sign of the difference $\Omega - A$.

The metric function A is indeed the angular velocity of a particle which is dragged along in the gravitational field of the star, as seen from a non-rotating observer at spatial infinity (∂_t), so that it has zero angular momentum relative to the axis, $p_\phi = 0$,

$$
\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{t\phi} p_t}{g^{tt} p_t} = \frac{g^{t\phi}}{g^{tt}} = \frac{-g_{t\phi}}{g_{\phi\phi}}; \qquad g_{t\phi} + \left(\frac{d\phi}{dt} \right) g_{\phi\phi} = 0; \qquad \frac{d\phi}{dt} = A \, .
$$

A is called *angular velocity of cumulative dragging* (shortly called *dragging rate*).[9,10] One of the purposes of this work is precisely to find appropriate bounds on the uniformly non-negative distribution of angular velocity, $\Omega \equiv \Omega(x^2, x^3) \geq 0$, of a slowly differentially rotating star, so that $\Omega - A \geq 0$ holds; from where the positivity of the angular momentum density (8) follows [Property (c) in Sec. V].

Observe, in the special relativistic limit [$g_{t\phi} \to 0$, using coordinates (x^2, x^3) which go at spatial infinity to the usual flat coordinates, cf. Sec. II B], if the fluid rotates uniformly with angular velocity Ω positive (negative), then the angular momentum density, Eq. (8), is uniformly positive (negative).

IV. SLOWLY DIFFERENTIALLY ROTATING STARS. FIRST ORDER PERTURBATIONS

By slow rotation we mean that the absolute value of the angular velocity is much smaller than the critical value $\Omega_{crit} \equiv (M/R^3)^{1/2}$ [taking units $c = G = 1$], where M is the total mass of the unperturbed (non-rotating) configuration, and R, its radius; $|\Omega(x^2, x^3)|/\Omega_{crit} \ll 1$. Thus, stars which rotate slowly can be studied by expanding the Einstein field equations for a fully relativistic differentially rotating star in powers of the dimensionless ratio

$$\frac{|\Omega_{max}|}{\Omega_{crit}} =: \mu\,, \tag{10}$$

where $|\Omega_{max}|$ is the maximum value of $|\Omega(x^2, x^3)|$ (at the interior of the star).

A. The metric and the energy density and pressure of the fluid

We assume that the star is slowly rotating, with angular velocity

$$\Omega(r, \theta) \equiv \Omega(x^2, x^3) = O(\mu)\,,$$

parameter μ given, e.g., by (10). Because the star (stationary in time and axially symmetric) rotates in the ϕ direction [(iv) of Sec. II A], a time reversal $(t \to -t)$ would change the sense of rotation, as well as an inversion in the ϕ direction $(\phi \to -\phi)$ would do. As a result, the metric coefficients H, Q and K [in (2)] and the energy density will not change sign under *one of* these inversions, whereas A will do. Therefore, an expansion of H, Q and K, as well as of the energy density, ε, and, hence, of the pressure, p, in powers of the angular velocity parameter μ will contain only even powers, while an expansion of the dragging rate, A, will have only odd ones. Hence, considering effects up to second order only the linear corrections in the dragging rate count. Indeed, the only first order $O(\mu)$ perturbation brought about by the rotation is the dragging of the inertial frames; the star is still spherical, because the "potential functions" which deform the shape of the star are $O(\mu^2)$. We shall keep here only the effects linear in the angular velocity. At first order, $O(\mu)$, the metric coefficients, and fluid energy density and pressure, are

$$\begin{aligned}
H &= H_0 + O(\mu^2) \\
Q &= Q_0 + O(\mu^2) \\
K &= K_0 + O(\mu^2) \\
\\
\varepsilon &= \varepsilon_0 + O(\mu^2) \\
p &= p_0 + O(\mu^2) \\
\\
\text{but} \;\; A &= \omega + O(\mu^3),
\end{aligned} \tag{11}$$

where H_0, Q_0 and K_0 are the coefficients of the unperturbed (non-rotating) configuration, and ω denotes the linear (first order) correction in μ of the dragging rate A, so that, from Eq. (9),

$$g_{t\phi} = -\omega \;(g_{\phi\phi})_0 + O(\mu^3)\,. \tag{12}$$

1. The (unperturbed) non-rotating configuration

The starting non-rotating equilibrium configuration is described by the spherically symmetric metric in the Schwarzschild form

$$\begin{aligned}
ds^2 &= -e^{\nu(r)}\,dt^2 + e^{\lambda(r)}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \\
&\equiv -{H_0}^2\,dt^2 + {Q_0}^2\,dr^2 + r^2{K_0}^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) \qquad [A_0 = 0]\,,
\end{aligned} \tag{13}$$

with $\lambda(r)$, or equivalently, the mass $m(r)$ interior to a given radial coordinate r, given by

$$1 - \frac{2m(r)}{r} = e^{-\lambda(r)}\,, \tag{14}$$

and $\nu(r)$, together with the pressure $p_0(r)$, and the energy density $\varepsilon_0(r)$, solutions of the system of equations of general relativistic hydrostatics, which for a non-rotating configuration are: the equation of hydrostatic equilibrium (Tolman-Oppenheimer-Volkoff equation),

$$\frac{dp_0}{dr}(r) = -\,\frac{[\varepsilon_0(r) + p_0(r)][m(r) + 4\pi r^3 p_0(r)]}{r^2[1 - 2m(r)/r]}\,, \tag{15}$$

the mass equation,

$$\frac{dm}{dr}(r) = 4\pi r^2 \varepsilon_0(r)\,, \tag{16}$$

and the source equation for ν,

$$\frac{d\nu}{dr}(r) = -\frac{2}{\varepsilon_0(r) + p_0(r)} \frac{dp_0}{dr}(r)\,, \tag{17}$$

with the initial boundary conditions

$$\begin{aligned} 0 < p_0(0) &= p_{0c} < \infty \text{ (central pressure)}, \\ m(0) &= 0\,, \quad \text{and} \\ \nu(0) &= \nu_c \text{ (constant fixed by the asymptotic condition at infinity)}, \end{aligned}$$

this being the prescription for the interior of the star, that is, inside the fluid, $r \leq R$, $R \equiv$ radius of the surface of the star, determined by $p_0(R) = 0$. Furthermore, we assume p_0 and ε_0 related to each other by a barotropic equation of state,

$$\varepsilon_0 = \varepsilon_0(p_0)\,, \tag{18}$$

$p_0 \mapsto \varepsilon_0(p_0)$ a bounded function on any closed interval, and satisfying the weak energy condition

$$\varepsilon_0 + p_0 \geq 0\,. \tag{19}$$

Observe, from Eqs. (15) and (19), p_0 is a decreasing function from the center, $r = 0$, to the star's surface, $r = R$; in particular, $p_0 \geq 0$ and attains its maximum value p_{0c} at the center.

In the exterior (vacuum) the geometry is described by the same line element (13), but with the metric function ν specified and related to λ by

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{2M}{r}, \quad \forall r > R\,, \tag{20}$$

where $M \equiv m(R)$ is the star's total mass.

Notice that this standard form of the non-rotating metric, (13), is the limit of zero rotation of the general rotating metric in spherical polar coordinates (2),

$$H \to H_0 \equiv e^{\nu/2}, \quad Q \to Q_0 \equiv e^{\lambda/2}, \quad K \to K_0 \equiv 1, \quad A \to A_0 \equiv 0\,;$$

from where the effect of the rotation can be seen as given by the term $d\phi - Adt$ in the place of $d\phi$.

Note, since at first order in the angular velocity there is still no effect on the pressure and on the energy density [cf. (11)], conditions (18) and (19) for the starting non-rotating configuration guarantee (vi) and (vii) of Sec. II A at first order.

B. Euler equation

It will be important to note that at first order the Ω-profile is not restricted by the field equations (through the Euler equation). That shows up in the following. Consider the first integral of the Euler equation (3), namely,

$$\int_0^{p(r,\theta)} \frac{d\bar{p}}{\varepsilon(\bar{p})+\bar{p}} + \frac{1}{2}\ln[(u^t)^{-2}]\bigg|_{(r,\theta)} + \int_{\Omega_0}^{\Omega(r,\theta)} \mathcal{F}(\Omega)\, d\Omega = \text{const.}\,, \tag{21}$$

where Eqs. (4) and (5) have been used, and Ω_0 is a given constant (changing the value of Ω_0 simply modifies the value of the constant on the right hand side). The first term in Eq. (21) is a function of the pressure, which is, to this approximation, a function of r, i.e. $O(1)$ with respect to the angular velocity; on the other hand,

$$(u^t)^{-2} = -(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}) = H^2 - K^2 r^2 \sin^2\theta\, (\Omega - A)^2 = O(1 - (\Omega - A)^2)\,,$$

so the second term is $O((\Omega - A)^2)$ and, hence, $O(\mu^2)$; also, since

$$u^t u_\phi = (u^t)^2 g_{\phi\phi}(\Omega - A) = \frac{K^2 r^2 \sin^2\theta\, (\Omega - A)}{H^2 - K^2 r^2 \sin^2\theta\, (\Omega - A)^2} = O(\Omega - A)\,,$$

$\mathcal{F}(\Omega) = u^t u_\phi = O(\Omega - A)$, thus, the third term is $O((\Omega - A)^2)$ as well, and, hence, $O(\mu^2)$. Consequently, to $O(\mu)$, the Euler equation reduces to its static (non-rotating) case, and indeed we have presumably already used it to get the starting unperturbed solution. Therefore, at this order in the angular velocity, Eq. (5) is no restriction on $\Omega(r, \theta)$.

C. The angular momentum density

Using the definition of ω, linear correction of the dragging rate, via the expansion of the metric coefficient $g_{t\phi}$, Eq. (12), and the metric coefficients of the non-rotating configuration [cf. (13)], we obtain the expansion for the angular momentum density (8)

$$\begin{aligned} T_\alpha{}^\beta \eta^\alpha n_\beta &= n_t T_\phi{}^t = n_t\, (\varepsilon + p)(u^t)^2 (g_{t\phi} + \Omega g_{\phi\phi}) \\ &= (n_t)_0\, (\varepsilon_0 + p_0)[(-g_{tt})_0]^{-1}[-\omega\ (g_{\phi\phi})_0 + \Omega\ (g_{\phi\phi})_0] + O(\mu^3) \\ &= e^{\nu/2}\, (\varepsilon_0 + p_0) e^{-\nu}\, r^2 \sin^2\theta\ (\Omega - \omega) + O(\mu^3) \\ &= (\varepsilon_0 + p_0)\, e^{-\nu/2}\, r^2 \sin^2\theta\ (\Omega - \omega) + O(\mu^3)\,. \end{aligned} \tag{22}$$

Thus showing also for the first order rotational perturbation that, since we are assuming the energy condition $\varepsilon_0 + p_0 \geq 0$, the sign of the angular momentum density (22) to $O(\mu)$ is determined by the sign of $\Omega - \omega$.

D. The rotational mass-energy

In Ref. 3 Hartle has derived the difference in total mass-energy, $M_{\rm rot}$, between a slowly and differentially rotating relativistic star and a non-rotating star with the same number of baryons and the same distribution of entropy, namely,

$$M_{\rm rot} = \frac{1}{2}\int_{\mathcal{I}} \Omega\, dJ + O(\mu^4)\,,$$

where dJ is the angular momentum of a fluid element in the star (to first order in the angular velocity), i.e., from (7),

$$dJ = T_\phi{}^t (-\mathrm{g})^{1/2} d^3x|_{O(\mu)}\,;$$

taking into account (22) and (13), we obtain an explicit expression for the expansion of $M_{\rm rot}$ in powers of the angular velocity parameter μ,

$$M_{\rm rot} = \frac{1}{2}\int_0^R dr \int_0^\pi d\theta\, 2\pi(\varepsilon_0 + p_0)\, r^4\, e^{(\lambda-\nu)/2}\, \sin^3\theta\, \Omega(\Omega - \omega) + O(\mu^4)\,. \tag{23}$$

E. The time-angle component of the Einstein equation

The $(t\phi)$ field equation component retaining only first order terms in the angular velocity (second order terms vanish), i.e., from Eq. (1),

$$R_\phi{}^t = 8\pi T_\phi{}^t + O(\mu^3)\,,$$

takes the form

$$\partial_r[r^4 j(r)\,\partial_r\omega] + \frac{r^2\,k(r)}{\sin^3\theta}\partial_\theta[\sin^3\theta\,\partial_\theta\omega] - 16\pi\, r^4 k(r)[\varepsilon_0(r) + p_0(r)]\,[\omega - \Omega] = 0\,, \tag{24}$$

where we have introduced the abbreviations

$$j(r) \equiv e^{-[\lambda(r)+\nu(r)]/2} \quad \text{and} \quad k(r) \equiv e^{[\lambda(r)-\nu(r)]/2}\,. \tag{25}$$

As outlined in Ref. 2, using the 0-order field equations, (14)-(17), it follows

$$4\pi\, r\,[\varepsilon_0(r) + p_0(r)]\, k(r) = -j'(r)\,, \tag{26}$$

(where $' \equiv d/dr$) which, substituted into Eq. (24), yields

$$\partial_r[r^4 j(r)\,\partial_r\omega] + \frac{r^2\,k(r)}{\sin^3\theta}\,\partial_\theta[\sin^3\theta\,\partial_\theta\omega] + 4\,r^3 j'(r)\,\omega \;= 4\,r^3 j'(r)\,\Omega(r,\theta)\,. \tag{27}$$

We write this differential equation for the dragging rate in the abbreviated form

$$\bar{L}\omega = -\Psi^2\Omega\,, \tag{28}$$

with the linear second order partial differential operator $\bar{L} \equiv \bar{L}_0 - \Psi^2$, where

$$\bar{L}_0\omega \;:=\; \frac{1}{r^4 j(r)}\partial_r[r^4 j(r)\,\partial_r\omega] + \frac{k(r)}{r^2 j(r)\sin^3\theta}\,\partial_\theta[\sin^3\theta\,\partial_\theta\omega] \quad \text{and} \tag{29}$$

$$\Psi^2(r) \;:=\; -\frac{4}{r}\frac{j'(r)}{j(r)} \;=\; 16\pi[\varepsilon_0(r) + p_0(r)]\,\frac{k(r)}{j(r)} \geq 0 \quad \forall r \geq 0\,. \tag{30}$$

Equation (26) has been used in (30), and the sign follows from the assumed energy condition (19), the functions j and k [cf. (25)] are always positive. Notice, $\Psi^2 \equiv 0$ in the exterior ($\forall r \in [R,\infty[$), where vacuum [$\varepsilon_0 = p_0 = 0$, cf. (iii) in Sec. II A] is considered.

Specifically, we are only interested in solutions $\omega \equiv \omega(r,\theta)$ of Eq. (28) in $[0,\infty[\times[0,\pi]$, which satisfy the *boundary conditions*

$$\omega \;\; \text{asymptotically flat} \;\; \left(\lim_{r\to\infty}\omega = 0\right), \tag{31}$$

$$\omega \;\; C^1\text{-regular on the axis of rotation,} \tag{32}$$

and a *matching condition*, namely, to be at least a class C^1 function on the surface of the star –which is spherical at first order rotational perturbations–

$$\omega(.,\theta) \;\; \text{class} \;\; C^1 \;\; \text{across} \;\; r = R\,. \tag{33}$$

Notice, (31) follows from our star model [condition (ii) in Sec. II A], and it can be easily seen that (33) follows from the equation itself, provided that $\omega(.,\theta)$ and $\Omega(.,\theta)$ are at least essentially bounded ($\in L^\infty$) —as has been assumed—, i.e. even if they have a jump discontinuity.

At the star's surface, $r = R$, higher regularity of $\omega(.,\theta)$ is not guaranteed by the equation, due to a jump discontinuity of the function Ψ^2 at this point. For this reason, we shall be considering (in the following section) *generalized* ($\in W^{1,2}$) solutions ω of Eq. (28) in the whole domain (interior and exterior).

"Coordinate change"

In order to avoid the coordinate singularity occurring on the axis in polar coordinates (r,θ), and wishing instead to have in the differential operator (29) a Lapacian in some higher dimension, we consider the following "change of coordinates". Firstly, we introduce *isotropic* cylindrical coordinates in the meridian plane,

$$(r,\theta) \mapsto (\,\rho := h(r)\sin\theta\ ,\ z := h(r)\cos\theta\,)\ \in \mathbb{R}_0^+ \times \mathbb{R}\,, \tag{34}$$

with the function h satisfying the following ordinary differential equation of first order with separated coefficients

$$\frac{h'(r)}{h(r)} = \frac{e^{\lambda(r)/2}}{r} \tag{35}$$

[which makes the coefficient of the crossed derivatives in the operator (29) after the change (34) to vanish], and the *boundary condition*

$$\lim_{r\to\infty} \frac{h(r)}{r} = 1\,, \tag{36}$$

i.e. so that the *isotropic* radius $h(r) \equiv \bar{r}$ approaches r at spatial infinity, because far away from the source we assume to have euclidean geometry. This leads us to the definition of the function [having $\omega(r,\theta)$]

$$w(\rho,z) := \omega\Big(h^{-1}(\sqrt{\rho^2+z^2}\,)\,,\ \arctan(\rho/z)\Big)\ , \tag{37}$$

or inversely, w such that

$$\omega(r,\theta) = w\,(\,h(r)\sin\theta\,,\ h(r)\cos\theta\,)\ .$$

Secondly, (in the spirit of Ref. 11) we define [with $w(\rho,z)$] the 5-*lift of* $w : \mathbb{R}_0^+\times\mathbb{R} \to \mathbb{R}$ in flat $\mathbb{R}^5$, axisymmetric around the x_5-axis, by

$$w \mapsto \tilde{\omega} \quad \text{such that} \quad \tilde{\omega}(\mathbf{x}) \equiv \tilde{\omega}(x_1,x_2,x_3,x_4,x_5) := w\big(\,\rho = (\textstyle\sum_{i=1}^4 x_i^2)^{1/2}\,,\ z = x_5\,\big)\ , \tag{38}$$

and, for every function $\tilde{\omega} : \mathbb{R}^5 \to \mathbb{R}$, the *meridional cut (in direction x_1) of* $\tilde{\omega}$

$$\tilde{\omega} \mapsto w \quad \text{such that} \quad w(\rho,z) := \tilde{\omega}(\rho,0,0,0,z)\,.$$

For axisymmetric functions these are isometric operations inverse to each other.[11] After considering the change of variable (34) with (37) in the differential operator $\bar{L}_0$ (29), we get [remember (25), $e^\lambda = k/j$]

$$\bar{L}_0\, w = \frac{e^{\lambda(r)} h(r)^2}{r^2} \Big\{ \partial_{\rho\rho} w + \partial_{zz} w + \frac{3}{\rho}\, \partial_\rho w + H(r) \frac{\rho\, \partial_\rho w + z\, \partial_z w}{h(r)} \Big\}\,, \tag{39}$$

where

$$H(r) := \frac{-e^{\frac{-\lambda(r)}{2}} \,[-6 + 6\, e^{\frac{\lambda(r)}{2}} + r\, \nu'(r)]}{2\, h(r)}\,. \tag{40}$$

But, through the 5-lift (38), the flat Laplacian in 5 dimensions of the "lifted" function $\tilde{\omega}$ gives exactly

$$\Delta\tilde{\omega} \;\equiv\; \sum_{i=1}^{5} \partial_{ii}\tilde{\omega} \;=\; \partial_{\rho\rho} w + \partial_{zz} w + \frac{3}{\rho}\, \partial_\rho w\,, \tag{41}$$

first three terms in the bracket of (39). Furthermore, as outlined in Ref. 11, n-lift and meridional cut (of *axisymmetric functions*) leave the regularity properties and the norm invariant; and *axisymmetric* operations, like multiplication, $\partial_{\bar{r}}$ $[\bar{r} \equiv h(r) = (\rho^2 + z^2)^{1/2} = (\sum_{i=1}^5 x_i^2)^{1/2}\,]$, and scalar product, commute with n-lift and meridional cut. In particular, in the fourth term in the bracket of (39) the factor

$$\rho\, \partial_\rho w + z\, \partial_z w \;=\; \partial_{\bar{r}}\tilde{\omega} \;=\; \sum_{i=1}^{5} x_i\, \partial_i \tilde{\omega}\,. \tag{42}$$

Therefore, substituting (41) and (42) into (39), Eq. (28) in the form $\bar{L}_0\tilde{\omega} = -\Psi^2(\tilde{\Omega} - \tilde{\omega})$ (with $\tilde{\Omega}$ defined from Ω as it was $\tilde{\omega}$ from ω, and $\Psi^2 = e^\lambda 16\pi[\varepsilon_0 + p_0]$) now writes

$$\bar{L}_0\, \tilde{\omega} \equiv \frac{e^{\lambda(r)} h(r)^2}{r^2} \Big\{ \Delta\tilde{\omega} + H(r) \frac{\sum_{i=1}^5 x_i\, \partial_i \tilde{\omega}}{h(r)} \Big\} = e^{\lambda(r)} \Big\{ -16\pi[\varepsilon_0(r) + p_0(r)][\tilde{\Omega} - \tilde{\omega}] \Big\}\,,$$

or, equivalently,[12]

$$\Delta\tilde{\omega} + H(r) \frac{\sum_{i=1}^5 x_i\, \partial_i \tilde{\omega}}{h(r)} = -16\pi \frac{r^2}{h(r)^2} [\varepsilon_0(r) + p_0(r)][\tilde{\Omega} - \tilde{\omega}]\,. \tag{43}$$

V. PROPERTIES

With the assumptions made in Sec. II A for this slowly rotating configuration [starting from a non-rotating one as described in Sec. IV A 1; particularly, satisfying the energy condition $\varepsilon_0 + p_0 \geq 0$], and considering only solutions of Eq. (28) satisfying the boundary and matching conditions (31), (32), and (33), the following results hold

Property (a) [Positiveness of the *dragging rate*]

If the distribution of angular velocity of the fluid is non-negative (and non-trivial), then the dragging rate *(to first order in the fluid angular velocity) is positive everywhere,*

$$\Omega \geq 0, \quad \Omega \not\equiv 0 \quad \Longrightarrow \quad \omega > 0\,.$$

Proof. We have seen in the former section that, using the coordinate change (34) and the 5-lift (38), Eq. (28) for ω is equivalent to Eq. (43) for $\tilde{\omega}$, which reads

$$L\tilde{\omega} := \Delta\tilde{\omega} + \sum_{i=1}^{5} H(r)\frac{x_i}{h(r)}\,\partial_i\tilde{\omega} - 16\pi[\varepsilon_0 + p_0]\frac{r^2}{h(r)^2}\tilde{\omega} = -16\pi[\varepsilon_0 + p_0]\frac{r^2}{h(r)^2}\tilde{\Omega}\,. \quad (44)$$

This equation may be obviously written *in divergence form*

$$L\tilde{\omega} \equiv \partial_i[a^{ij}(\mathbf{x})\partial_j\tilde{\omega} + a^i(\mathbf{x})\,\tilde{\omega}] + b^i(\mathbf{x})\,\partial_i\tilde{\omega} + c(\mathbf{x})\,\tilde{\omega} \;=\; g(\mathbf{x})$$

(where repeated indices denote summation over the index), with the coefficients

$$\begin{aligned} a^{ij}(\mathbf{x}) &\equiv \delta_{ij} \quad (= 1 \text{ if } i = j, \text{ and } = 0 \text{ otherwise})\,, \\ a^i(\mathbf{x}) &\equiv 0\,, \\ b^i(\mathbf{x}) &= H(r)\frac{x_i}{h(r)} \qquad (\forall\, i, j \in \{1, \ldots, 5\}), \quad \text{and} \qquad (45) \\ c(\mathbf{x}) &= -16\pi[\varepsilon_0(r) + p_0(r)]\frac{r^2}{h(r)^2} \quad (\leq 0)\,, \\ \text{and} \quad g(\mathbf{x}) &= c(\mathbf{x})\,\tilde{\Omega}(\mathbf{x})\,. \qquad (46) \end{aligned}$$

The isotropic radius $\bar{r} \equiv h(r)$ is Gaussian coordinate with respect to the star's surface, $\bar{r} = h(R)$, and, thus, $\tilde{\omega}$ is at least class C^1 across this surface; therefore, $\tilde{\omega} \in C^1(\mathbb{R}^5)$. Thus, considering the domain G defined by a ball in $\mathbb{R}^5$ centered at the origin of coordinates $\mathbf{x} = \mathbf{0}$ ($\bar{r} = 0$) and of arbitrary large radius σ,

$$G := \mathcal{B}_\sigma(\mathbf{0}) \subset \mathbb{R}^5\,, \qquad (47)$$

the function $\tilde{\omega}$ and its (first) derivatives (continuous in $\mathbb{R}^5$) are 2-integrable in G, i.e. $\tilde{\omega} \in W^{1,2}(G) \cap C^1(G)$, and Eq. (43) is satisfied in G in a generalized sense (cf. Appendix B).

Notice, whenever $\tilde{\Omega} \geq 0$, we have, by (46), $g \leq 0$ (because $c \leq 0$), and, hence, $L\tilde{\omega} \leq 0$, specifically $\tilde{\omega}$ is a generalized supersolution relative to the operator L, in (44), and the domain G, (47). We look at the requirements for a minimum principle to be applied (Appendix B). The Laplacian operator is obviously strictly elliptic, and the coefficients (45) are measurable and bounded functions on G, this shows up in the following: the mapping $r \mapsto \frac{r}{h(r)}$ is bounded from above and below everywhere in $[0, \infty[$ (Appendix A); $\varepsilon_0 + p_0$ is also bounded, since p_0 is bounded and $p_0 \mapsto \varepsilon_0(p_0)$ is bounded in any closed interval (Sec. IV A 1); consequently, the coefficient c is bounded (from above and below); the coefficients of the first derivatives, b^i, are also bounded (from above and below), because the function H is bounded everywhere (Appendix A) and since $(\forall i = 1, \ldots, 5)$ ${x_i}^2 \leq \sum_{j=1}^{5} {x_j}^2 = h(r)^2$, we have $[x_i/h(r)]^2 \leq 1$. Thus, all conditions of a minimum principle for generalized supersolutions relative to the differential operator L and the domain G hold, and, as a result of the *weak* minimum principle (Theorem 1 in Appendix B), we have

$$\inf_G \tilde{\omega} \geq \inf_{\partial G} \tilde{\omega}^- \quad [\tilde{\omega}^- \equiv \min(\tilde{\omega}, 0)\,]\,. \tag{48}$$

But, since the radius of the ball G is arbitrary, we can make it sufficiently large ($\sigma \to \infty$) so that, by asymptotic flatness $[\lim_{h(r)\to\infty} \tilde{\omega} = 0$, from condition (31) and $\lim_{r\to\infty} \frac{h(r)}{r} = 1]$, $\tilde{\omega}$ is arbitrary small at ∂G, following, from (48), $\tilde{\omega} \geq 0$. Actually, the positivity is strict, because if $\tilde{\omega}(\mathbf{x_0}) = 0$ for some $\mathbf{x_0} \in G$ (interior point), then $\tilde{\omega}(\mathbf{x_0}) = \min \tilde{\omega}$ [since $\tilde{\omega} \geq 0$], and, by the *strong* minimum principle (Theorem 2 in Appendix B), $\tilde{\omega}$ would be constant in G; in this case, $\tilde{\omega} \equiv$ const. $= 0$ in G (i.e. everywhere); but $\tilde{\omega} \equiv 0$ yields, by Eq. (44), $\tilde{\Omega} \equiv 0$, or, equivalently, $\Omega \equiv 0$, and we are assuming that Ω is non-trivial. We conclude then $\omega > 0$ everywhere. $\square$

Property (b)

Suppose we perturb the non-rotating configuration (in particular, with a given equation of state) with two (small) different distributions of angular velocity, Ω_1 *and* Ω_2*, and integrate Eq.* (28) *to obtain their respective solutions for the dragging rate,* ω_1 *and* ω_2*, then*

$$\Omega_1 \geq \Omega_2, \quad \Omega_1 \not\equiv \Omega_2 \quad \Longrightarrow \quad \omega_1 > \omega_2 \,.$$

Proof. This follows form the linearity of Eq. (28) and Property (a). $\square$

We are already in position to get a result about the positiveness of the difference $\Omega - \omega$, and, hence, of the *angular momentum density* (22). However, in order to first do this more specific and concrete, we shall make use of a property for the particular case of rigid rotation (RR), which can be found in Ref. 2, Sec. IV.

Property RR

For the slowly rotating configuration,

$$\begin{array}{ccl} & & \omega(r,\theta) = \omega(r) \\ \Omega(r,\theta) \equiv \ \text{const.} \equiv \widehat{\Omega} > 0 & \Longrightarrow & 0 < \omega(r) < \widehat{\Omega} \quad \textit{in } [0,R] \\ (\textit{in } [0,R]\times[0,\pi]\,) & & \omega > 0 \textit{ in } [0,\infty[\,,\ \omega' < 0 \textit{ in }]0,\infty[\,,\ \omega'(0) = 0\,. \end{array}$$

Property (c) [Positiveness of the *angular momentum density*]

For the slowly rotating configuration, with a given equation of state, its dragging rate ω *will satisfy*

$$\omega(r,\theta) < \Omega(r,\theta)$$

if $\Omega \equiv \Omega(r,\theta)$ (≥ 0) *is bounded in the form*

$$\underline{\Omega}(\overline{\Omega}) \equiv \underline{\Omega} \ \leq \ \Omega(r,\theta) \ \leq \ \overline{\Omega}\,,$$

(in $[0,R]\times[0,\pi]$*) where* $\overline{\Omega}$ *is an arbitrary positive constant* $0 < \overline{\Omega}$ $(\ll \Omega_{crit})$ *(Sec. IV), and* $\underline{\Omega} = \overline{\omega}(0)$, $\overline{\omega}$ *solution of Eq.* (28) *with* $\overline{\Omega}(r,\theta) = \text{const.} = \overline{\Omega}$*, and with the same* 0*-order coefficients (same starting non-rotating configuration) as the ones considered for our slowly rotating configuration, in particular with the same equation of state; or, more generally, if (with that notation)*

$$\overline{\omega}(r) \ \leq \ \Omega(r,\theta) \ \leq \ \overline{\Omega}\,.$$

Notice, $\omega < \Omega$ *means that the* angular momentum density *to first order in the fluid angular velocity,* (22)*, of this configuration [with the energy condition* (19)*] is* ≥ 0*, vanishing on the axis.*

[Remarkably, the upper bound required on Ω is not restrictive, because for Ω continuous, Ω is essentially bounded ($\in L^\infty$) there, and $\Omega/\|\Omega\|_\infty \leq 1$.]

Proof. We give a practical method of construction in two steps:

1st step: Consider $\overline{\Omega}(r,\theta) := \overline{\Omega} = \text{const.} > 0$, and solve the corresponding Eq. (28) for $\overline{\omega}$. Then, by Property RR, the solution satisfies

$$\overline{\omega}(r,\theta) = \overline{\omega}(r)\,, \tag{49}$$

$$0 < \overline{\omega}(r) < \overline{\Omega} \quad \text{in } [0,R]\,, \tag{50}$$

$$\text{and} \quad \overline{\omega} > 0 \,\text{in } [0,\infty[\,, \quad \overline{\omega}' < 0 \text{ in }]0,\infty[\,, \quad \overline{\omega}'(0) = 0\,. \tag{51}$$

2nd step: Consider a slowly rotating configuration starting from the same non-rotating configuration (as in the first step) with a fluid angular velocity distribution $\Omega(r,\theta)$ such that

$$\overline{\omega}(0) =: \underline{\Omega} \ \leq \ \Omega(r,\theta) \ \leq \ \overline{\Omega}\,. \tag{52}$$

Observe, we are always allowed to do this because of (50). Or, more generally, such that $\overline{\omega}(r) \leq \Omega(r,\theta) \leq \overline{\Omega}$; notice, from (51), $\overline{\omega}$ is a (positive) decreasing function; in particular, $\overline{\omega}(0) \geq \overline{\omega}(r) > 0, \quad \forall r \in [0,\infty[$.

From the second inequality in (52), i.e. from $\Omega(r,\theta) \leq \overline{\Omega}$, and, since we have the same starting unperturbed configuration (same 0-oder coefficients) for these both slowly rotating configurations, it follows, by Property (b), that their corresponding solutions [of Eq. (28)] satisfy

$$\omega(r,\theta) < \overline{\omega}(r)\,, \tag{53}$$

where we have used (49). On the other hand, the first inequality in (52), and (51) yield

$$\overline{\omega}(r) \leq \overline{\omega}(0) := \underline{\Omega} \ \leq \ \Omega(r,\theta)\,, \tag{54}$$

and consequently, from (53) and (54),

$$\omega(r,\theta) \ < \ \Omega(r,\theta)\,.$$

□

Remark. Notice, the same argument also assures that, given a slowly rotating configuration with $\overline{\Omega}(r,\theta)$ such that the corresponding dragging rate $\overline{\omega}(r,\theta) < \overline{\Omega}(r,\theta)$, we shall have the same positivity result [Property (c)] for any slowly rotating configuration, starting from the same unperturbed configuration (in particular, with the same equation of state), with an angular velocity distribution $\Omega(r,\theta)$ such that

$$\underline{\Omega}(r,\theta) := \overline{\omega}(r,\theta) \ \leq \ \Omega(r,\theta) \ \leq \ \overline{\Omega}(r,\theta)\,,$$

because we obtain, form the last inequality and Property (b), $\omega(r,\theta) < \overline{\omega}(r,\theta)$, and, hence, $\omega(r,\theta) < \Omega(r,\theta)$.

Series expansion. M_{rot}

Before we prove next property, we first stress that, since Ω and ω transform like vectors under rotation, Eq. (27) may be separated by expanding them as

$$\Omega(r,\theta) \ \equiv \ \Omega(r,x) \sim \sum_{l=1}^{\infty} \Omega_l(r)\, y_l(x) \ \text{ and} \tag{55}$$

$$\omega(r,\theta) \ \equiv \ \omega(r,x) \sim \sum_{l=1}^{\infty} \omega_l(r)\, y_l(x)\,, \tag{56}$$

with the change of variable $\theta \mapsto x := \cos\theta$, and

$$y_l(x) := \frac{d\mathcal{P}_l}{dx} \quad \forall x \in [-1,1] \ \ (\theta \in [0,\pi]), \quad \mathcal{P}_l \equiv \text{Legendre polynomial of degree } l\,. \tag{57}$$

Then the equation for ω_l takes the form

$$\frac{d}{dr}[r^4 j(r)\ \omega_l'] + [4\,r^3 j'(r) - r^2\,k(r)\,\lambda_l]\ \omega_l = 4\,r^3 j'(r)\ \Omega_l(r)\,, \tag{58}$$

with $\lambda_l := l(l+1) - 2,\ \ l \in \mathbb{N},\ l \neq 0$, and j and k defined in (25).

From conditions (31) and (32) on ω, we have the respective boundary conditions on ω_l

$$\lim_{r\to\infty} \omega_l(r) = 0\,, \tag{59}$$

$$\omega_l \ \ C^1\text{-regular at the origin,} \tag{60}$$

and, from (33), the matching condition

$$\omega_l \ \ \text{class} \ \ C^1 \ \ \text{across} \ \ r = R\,. \tag{61}$$

In Sec. IV D an explicit expression for the expansion of the rotational mass-energy $M_{\rm rot}$ in powers or the angular velocity parameter was obtained (23), or, using Eq. (26),

$$M_{\rm rot} = -\frac{1}{4}\int_0^R dr\, r^3\, j' \int_0^\pi d\theta \sin^3\theta\, \Omega(\Omega-\omega) + O(\mu^4)\,. \tag{62}$$

Using the series expansions of Ω and ω, (55) and (56), and the fact that the system $\{y_l\}_{l=1}^\infty$ is orthogonal in the Hilbert space $L^2_\rho([-1,1])$, with respect to the weight function $\rho(x) := 1-x^2$, $x\in[-1,1]$, and have norm $\|y_l\|_\rho{}^2 = 2l(l+1)/(2l+1)$, the integral over θ in (62) may be expressed as the sum

$$\int_0^\pi d\theta \sin^3\theta\, \Omega(r,\theta)\,[\Omega(r,\theta)-\omega(r,\theta)] = \sum_{l=1}^\infty \frac{2l(l+1)}{2l+1}\Omega_l(r)\,[\Omega_l(r)-\omega_l(r)]\ , \tag{63}$$

and, consequently, the rotational mass-energy (62) can be expressed as a sum of integrals (over r)

$$M_{\rm rot} = \sum_{l=1}^\infty \frac{l(l+1)}{2(2l+1)}\, M_l\ +\ O(\mu^4)\,, \tag{64}$$

$$\text{with}\quad M_l := \int_0^R f^2(r)\,\Omega_l(r)\,[\Omega_l(r)-\omega_l(r)]\ dr\,, \qquad f^2(r) := -r^3\, j'(r)\ \geq 0\,. \tag{65}$$

Property (d) [Positivity and upper bound on the *rotational energy* $M_{\rm rot}$]

We consider Eq. (58), which can be written

$$\frac{d}{dr}(r^4\, j\,\omega_l') - r^2\, k\,\lambda_l\,\omega_l = -4\, f^2\,(\Omega_l-\omega_l)\,. \tag{66}$$

The main observation is that, multiplying both sides of Eq. (66) by ω_l, integrating from $r=0$ to $r=\infty$, and taking into account that $f^2 = -r^3\, j' = 4\pi\, r^4\,(\varepsilon_0+p_0)\, k \equiv 0\ \ \forall r>R$,

$$\int_0^\infty \left[\frac{d}{dr}(r^4\, j\,\omega_l')\,\omega_l - r^2\, k\,\lambda_l\,\omega_l{}^2\right] dr = -4\int_0^R f^2\,\omega_l\,(\Omega_l-\omega_l)\, dr$$

[note, the integral on the left hand side converges, because an asymptotically flat (59) solution of Eq. (58) must behave as $r\to\infty\quad \omega_l = O(r^{-l-2})$, and $\omega_l' = O(r^{-l-3})$, $l\geq 1$]; and, after integrating once by parts the first term on the left hand side,

$$r^4\, j\,\omega_l'\,\omega_l\big|_0^\infty - \int_0^\infty \left[r^4\, j\,(\omega_l')^2 + r^2\, k\,\lambda_l\,\omega_l{}^2\right] dr = -4\int_0^R f^2\,\omega_l\,(\Omega_l-\omega_l)\, dr\,.$$

The first term vanishes because ω_l falls off rapidly enough at $r \to \infty$, and the second term (minus the integral on the left hand side) is non-positive (since j and k are always positive), therefore

$$\int_0^R f^2 \, \omega_l \, (\Omega_l - \omega_l) \, dr \; \geq 0 \, . \tag{67}$$

Using a few simple linear algebra calculations [including the Cauchy-Schwarz inequality for the bilinear form $\langle u, v \rangle_f := \int_0^R f^2(r) \, u(r) \, v(r) \, dr \quad u, v \in C^0([0, R])$], the former inequality (67) yields [see Ref. 1, Sec. V] $0 \leq M_l \leq \int_0^R f^2 \, {\Omega_l}^2 \, dr$, which gives respective bounds on $M_{\rm rot}$ [cf. (64)],

$$0 \leq M_{\rm rot} \leq \sum_{l=1}^{\infty} \frac{l(l+1)}{2(2l+1)} \int_0^R f^2(r) \, {\Omega_l}^2(r) \, dr \; + \, O(\mu^4) \, ,$$

or, writing the sum as integral over θ [as in (63)],

$$0 \leq M_{\rm rot} \leq \frac{1}{4} \int_0^R dr \, f^2(r) \int_0^\pi d\theta \, \sin^3\theta \, [\Omega(r, \theta)]^2 \; + \, O(\mu^4) \, , \tag{68}$$

where $f^2 := -r^3 \, j' = 4\pi \, r^4 \, (\varepsilon_0 + p_0) e^{(\lambda - \nu)/2}$. Additionally, $\int_0^R f^2 \, {\omega_l}^2 \, dr \leq \int_0^R f^2 \, {\Omega_l}^2 \, dr$ also follows from (67), yielding the "mean values" inequality (in full general)

$$\int_0^R dr \, f^2(r) \int_0^\pi d\theta \, \sin^3\theta \, [\omega(r, \theta)]^2 \; \leq \int_0^R dr \, f^2(r) \int_0^\pi d\theta \, \sin^3\theta \, [\Omega(r, \theta)]^2 \; + \, O(\mu^4) \, . \tag{69}$$

□

VI. CONCLUDING REMARKS

Summing up, it has been seen that relativistic stars rotating slowly and differentially, with a non-negative (and non-trivial) angular velocity distribution, $\Omega(x_2, x_3) \geq 0$ ($\not\equiv 0$), and satisfying the energy condition $\varepsilon + p \geq 0$, have positive *rate of rotational dragging* $\omega > 0$ [Property (a) in Sec. V]; and a restriction on the amplitude of the Ω-profile assures also the positivity of the difference $\Omega - \omega$ and, hence, of the *angular momentum density*, this later vanishing on the axis, [Property (c)]. We also observe that, the *rotational mass-energy*, [from Property (d)] non-negative and (as expected) "increased" by a (slow) angular velocity of the fluid, Ω, is "decreased" by the dragging effect (over what it would be is this effect were neglected), i.e. is decreasing with respect to dragging rate, ω, despite of [as shown in Property (b)] ω

being an "increasing function" of Ω. Property (b) and, hence, also Property (c) are based on the linearity of the time-angle field equation component to first order in the fluid angular velocity. In the general differentially rotating case, i.e. outside the slow rotation limit, the rotation profile Ω cannot be freely chosen, but is restricted by the integrability condition of the Euler equation, i.e. by Eq. (5). This makes unlikely a generalization of Property (c) outside the slow rotation limit, other than in the form given in Ref. 1, Sec. IV B.

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APPENDIX A: Boundedness of some functions in $[0,\infty[\ni r$

The ratio *radius - isotropic radius* $\chi(r) := \frac{r}{h(r)}$

We have

$$\frac{h'(r)}{h(r)} = \frac{e^{\lambda(r)/2}}{r}\,, \tag{A1}$$

with

$$e^{-\lambda(r)} = 1 - \frac{2\,m(r)}{r}\,, \tag{A2}$$

where

$$m(r) := \begin{cases} 4\pi\int_0^r \varepsilon_0(s)s^2\,ds & : \quad r\in[0,R] \\ M \equiv 4\pi\int_0^R \varepsilon_0(s)s^2\,ds & : \quad r\in]R,\infty[\end{cases}\,, \tag{A3}$$

if we denote the stellar radius of the static model by $R > 0$. As we start with a (physically) regular (i.e. non-collapsed) static solution, we assume that $2m(r) < r$ (for all $r \in]0,R[$), and $2M < R$.

Integrating Eq. (A1) and using Eq. (A2) we get

$$h(r) = h(R)\,\exp\left(\int_R^r \frac{ds}{\sqrt{s[s-2m(s)]}}\right)\,. \tag{A4}$$

[Note, the constant $h(R)$ is determined by the asymptotic condition

$$\lim_{r\to\infty}\frac{h(r)}{r} = 1\,;$$

see below.] Let

$$g(r) := \int_R^r \frac{ds}{\sqrt{s[s-2m(s)]}} \quad \forall r > 0\,. \tag{A5}$$

With this definition the solution, (A4), now writes

$$h(r) = h(R)\,\exp\left(g(r)\right)\,. \tag{A6}$$

Due to the assumptions made for m, g is obviously a continuous function in the open interval $]0,\infty[$; consequently, by (A6), h is also a continuous function there, and, in particular, $h(r)$ cannot be zero in $]0,\infty[$ (unless $h(R)=0$, however this would contradict asymptotic flatness); therefore $\chi : r \mapsto \chi(r) := \frac{r}{h(r)}$ is continuous in $]0,\infty[$ as well. Choose an $\epsilon \in]0,R[$ and an $\epsilon' \in]R,\infty[$, then χ is bounded below and above in the interval $[\epsilon,\epsilon']$ (where the upper and lower bound depend on the selected ϵ and ϵ', of course). Let us now consider the intervals $[0,\epsilon]$ and $[\epsilon',\infty[$ separately:

On $[\epsilon',\infty[$: We have

$$0 \leq m(r) \equiv M\,.$$

Then

$$\frac{1}{s} \leq \frac{1}{\sqrt{s[s-2m(s)]}} \equiv \frac{1}{\sqrt{s[s-2M]}} \quad \forall r \in [\epsilon',\infty]\,.$$

As in this interval $r \geq R$, we find, with Eq. (A5),

$$\ln\left(\frac{r}{R}\right) = \int_R^r \frac{ds}{s} \leq g(r) \equiv \int_R^r \frac{ds}{\sqrt{s[s-2M]}} = 2\ln\left(\frac{\sqrt{r}+\sqrt{r-2M}}{\sqrt{R}+\sqrt{R-2M}}\right)\,.$$

Inserting it into Eq. (A6), yields (since exp is a monotonically increasing function)

$$\frac{h(R)}{R}\,r \leq h(r) \equiv h(R)\left(\frac{\sqrt{r}+\sqrt{r-2M}}{\sqrt{R}+\sqrt{R-2M}}\right)^2 \leq \frac{4h(R)}{\left(\sqrt{R}+\sqrt{R-2M}\right)^2}\,r\,. \tag{A7}$$

Note especially that $\lim_{r\to\infty}\frac{h(r)}{r} = \frac{4h(R)}{\left(\sqrt{R}+\sqrt{R-2M}\right)^2}$, but, by asymptotic flatness, $\lim_{r\to\infty}\frac{h(r)}{r} = 1$; therefore $h(R) = \frac{1}{4}\left(\sqrt{R}+\sqrt{R-2M}\right)^2 > 0$. Thus, $h(R) > 0$, and, from Eq. (A7),

$$0 < \frac{\left(\sqrt{R}+\sqrt{R-2M}\right)^2}{4h(R)} \leq \chi(r) \leq \frac{R}{h(R)} < \infty \quad \forall r \in [\epsilon',\infty[\,. \tag{A8}$$

On $[0,\epsilon]$**:** We have

$$0 \le m(r) = 4\pi \int_0^r \varepsilon_0(s) s^2\, ds \le \frac{4\pi}{3}\hat{\varepsilon}_0 r^3 =: \frac{c_0}{2} r^3\,,$$

where $\hat{\varepsilon}_0 := \sup_{r\in[0,R]} \varepsilon_0(r) > 0$. Next choose $\epsilon > 0$, such that $1 - c_0 r^2 > 0$ on $[0,\epsilon]$ [e.g., $\epsilon := (2\sqrt{c_0})^{-1}$]. Then

$$\frac{1}{s} \le \frac{1}{\sqrt{s[s-2m(s)]}} \le \frac{1}{s\sqrt{1-c_0 s^2}} \quad \forall r \in [0,\epsilon]\,,$$

and, since in this interval $r \le R$, we find, with Eq. (A5),

$$\ln\left(\frac{R}{r}\right) = \int_r^R \frac{ds}{s} \le -g(r) \le \int_r^R \frac{ds}{s\sqrt{1-c_0 s^2}} = \ln\left(\frac{R}{r}\right) + \ln\left(\frac{1+\sqrt{1-c_0 r^2}}{1+\sqrt{1-c_0 R^2}}\right)\,.$$

Again, inserting it into Eq. (A6), yields

$$\frac{h(R)}{R} r \ge h(r) \ge \frac{h(R)}{R}\frac{1+\sqrt{1-c_0 R^2}}{1+\sqrt{1-c_0 r^2}}\, r \ge \frac{h(R)\left(1+\sqrt{1-c_0 R^2}\right)}{2R}\, r\,,$$

and, hence,

$$0 < \frac{R}{h(R)} \le \chi(r) \le \frac{2R}{h(R)\left(1+\sqrt{1-c_0 R^2}\right)} < \infty \quad \forall r \in [0,\epsilon]\,. \tag{A9}$$

We can therefore conclude that, since $\mathbb{R}_0^+ = [0,\epsilon] \cup [\epsilon,\epsilon'] \cup [\epsilon',\infty[$ and χ is bounded (from above and below) in each of these subintervals, χ is bounded (from above and below) in $\mathbb{R}_0^+$. □

The function H

We have

$$H(r) := \frac{-e^{\frac{-\lambda(r)}{2}}\left[-6 + 6\, e^{\frac{\lambda(r)}{2}} + r\,\nu'(r)\right]}{2\,h(r)}\,,$$

and

$$\begin{aligned} e^{-\lambda(r)/2} &= \sqrt{1-\frac{2m(r)}{r}} \\ r\nu'(r) &= \frac{2m(r) + 8\pi r^3 p_0(r)}{r - 2m(r)} = \frac{\frac{2m(r)}{r} + 8\pi r^2 p_0(r)}{1 - \frac{2m(r)}{r}}\,. \end{aligned}$$

Thus,

$$H(r) = \frac{1}{2h(r)}\left\{6\left[\sqrt{1-\frac{2m(r)}{r}} - 1\right] - \left[\frac{2m(r)}{r} + 8\pi r^2 p_0(r)\right]\left(\sqrt{1-\frac{2m(r)}{r}}\right)^{-1}\right\}. \tag{A10}$$

Using the Cauchy-Schwarz inequality in (A10), and the following estimates in $r \in [0, \epsilon]$, for some $\epsilon \in]0, R[$ small, (see former section in Appendix A)

$$0 \le m(r) \le \frac{c_0}{2} r^3 \tag{A11}$$

$$0 \le \sqrt{1-x} \le 1 - \frac{x}{2} \quad \forall x \in [0,1] \tag{A12}$$

$$0 \le c_1 r \le h(r) \le c_2 r \tag{A13}$$

$$0 \le p_0(r) \le \hat{p}_0 := \sup_{r \in [0,R]} p_0(r) \quad \forall r \ge 0\,, \tag{A14}$$

where the constants $c_i\ _{(i=0,\ldots,2)}$ are all strictly positive (and finite), we get

$$\begin{aligned} |H(r)| &\le \frac{1}{2h(r)} \left\{ 6 \left| \sqrt{1 - \frac{2m(r)}{r}} - 1 \right| + \left| \frac{2m(r)}{r} + 8\pi r^2 p_0(r) \right| \left(\sqrt{1 - \frac{2m(r)}{r}} \right)^{-1} \right\} \\ &\le \frac{1}{2c_1 r} \left\{ 6 \left[\frac{c_0}{2} r^2 \right] + \left[c_0 r^2 + 8\pi \hat{p}_0 r^2 \right] (\sqrt{1 - c_0 \epsilon^2})^{-1} \right\} \\ &=: \frac{c_3 r^2}{c_1 r} =: c_4 r\,, \end{aligned} \tag{A15}$$

with $0 < c_3,\ c_4 < \infty$. Therefore H is bounded in $[0, \epsilon]$. [Especially, due to Eq. (A15), $H(0) = 0$.] And, since, by Eq. (A10), H is also continuous in the open interval $]0, \infty[$ and $\lim_{r\to\infty} H(r) = 0$ [because $\lim_{r\to\infty} \frac{h(r)}{r} = 1$], H is bounded everywhere in $[0, \infty[$.
$\square$

APPENDIX B: The minimum principle for generalized supersolutions

Consider in a domain (open and connected set) $G \subset \mathbb{R}^n \quad (n \ge 2)$ the differential operator with principal part of divergence form, defined by

$$Lu = \partial_i [a_{ij}(x) \partial_j u + a_i(x)\, u] + b_i(x)\, \partial_i u + c(x)\, u\,,$$

with $a_{ij} = a_{ji}$. Notice, an operator L of the general form $Lu = \tilde{a}_{ij}(x) \partial_{ij} u + \tilde{b}_i(x) \partial_i u + \tilde{c}(x) u$ may be written in divergence form provided its principal coefficients $\tilde{a}_{ij}$ are differentiable. If furthermore the $\tilde{a}_{ij}$ are constants, then even with coinciding coefficients ($a_{ij} = \tilde{a}_{ij}$, $b_i = \tilde{b}_i$, $c = \tilde{c}$) and $a_i \equiv 0$. Let us assume that

1. L is strictly elliptic in G, i.e. $\exists$ a constant $\lambda > 0$ such that $\lambda \le$ the minimum eigenvalue of the principal coefficient matrix $[a_{ij}(x)]$,

$$\lambda\, |y|^2 \le a_{ij}(x)\, y_i y_j \quad \forall y \in \mathbb{R}^n, \ \ \forall x \in G\,; \tag{B1}$$

2. a_{ij}, a_i, b_i, and c are measurable and bounded functions in G,

$$|a_{ij}| < \infty, \quad |a_i| < \infty, \quad |b_i| < \infty, \quad |c| < \infty \ \text{ in } G \quad (i, j \in \{1, \dots, n\}). \tag{B2}$$

By definition, for a function u which is only assumed to be *weakly differentiable* and such that the functions $a_{ij}\partial_j u + a_i u$ and $b_i \partial_i u + cu$, $i = 1, \dots, n$ are locally integrable [in particular, for u belonging to the Sobolev space $W^{1,2}(G)$], u is said to satisfy $Lu = g$ in G *in a generalized (or weak) sense* (g also a locally integrable function in G) if it satisfies

$$\begin{aligned} \mathcal{L}(u, \varphi; G) \ &:= \ \int_G \{(a_{ij}\partial_j u + a_i u)\partial_i \varphi - (b_i \partial_i u + cu)\varphi\} dx \\ &= \ -\int_G g\, \varphi\, dx, \qquad \forall \varphi \geq 0 \quad \varphi \in C_c^1(G) \end{aligned}$$

[where $C_c^1(G)$ is the set of functions in $C^1(G)$ with compact support in G].

Notice, u is *generalized supersolution* relative to a differential operator L and the domain G (i.e. satisfies $Lu \leq 0$ in G in a generalized sense) if it satisfies $\mathcal{L}(u, \varphi; G) \geq 0$, $\forall \varphi \geq 0 \quad \varphi \in C_c^1(G)$.

Theorem 1: (weak minimum principle)
Let $u \in W^{1,2}(G)$, G a bounded domain, satisfy $Lu \leq 0$ in G in a generalized sense with

$$\int_G (c\varphi - a^i \partial_i \varphi)\, dx \leq 0, \quad \forall \varphi \geq 0 \quad \varphi \in C_c^1(G). \tag{B3}$$

and conditions (B1) *and* (B2) *above,*
then

$$\min_{\overline{G}} u \geq \min_{\partial G} u^- \qquad [\, u^- \equiv \min(u, 0)\,].$$

(A proof of this theorem can be found in Ref. 13, Theorem 8.1.)

Theorem 2: (strong minimum principle)
Let $u \in W^{1,2}(G) \cap C^0(G)$ satisfy $Lu \leq 0$ in G in a generalized sense, with the operator L satisfying conditions (B1), (B2), *and* (B3),
then u cannot achieve a non-positive minimum in the interior of G, unless $u \equiv$ const.

(A proof of this theorem can be found in Ref. 13, Theorem 8.19.) Note that the weak minimum principle, Theorem 1, for $C^0(G)$ supersolutions is a direct consequence.

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[12] Indeed, the first order terms (in the fluid angular velocity) of the general metric, as given in Ref. 1, yield (omitting here the '$\tilde{\ }$' symbol for all 5-*lifted* functions) $H(r) = (3\,\partial_{\bar{r}}B - 4\,\partial_{\bar{r}}U)|_{h(r)}$, where $\bar{r} = h(r)$, given by Eq. (35). [This can be seen by a straightforward calculation, using $e^{2U(h(r))} = e^{\nu(r)}$, $h(r)^2\, e^{2[B(h(r))-U(h(r))]} = r^2$, and their differentiations with respect to r.] And since [from $h(r) = (\sum_{i=1}^{5} x_i^2)^{1/2}$] $\partial_i h(r) = x_i\, h(r)^{-1}$, we have $H(r)\, x_i\, h(r)^{-1} = (3\,\partial_i B - 4\,\partial_i U)|_{O(\Omega)}$, and, hence, $\langle 3\,\nabla B - 4\,\nabla U, \nabla\omega\rangle|_{O(\Omega)}$ as second term in Eq. (43). Also, since to first order [spherical; $K = B$] $e^{2K}N^{-1} = e^{2(B-U)} = r^2 h^{-2}$, the coefficient of the right hand side of Eq. (43) is actually $-\Psi(r)^2\, e^{-\lambda(r)} r^2 h(r)^{-2} = -\psi^2|_{O(\Omega)}$, in the notation of Ref. 1.

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Qualitative behavior of the dragging potential in slowly and differentially rotating stars

I. SERIES EXPANSIONS OF ω AND Ω

As outlined in Ref. 1 [**Paper II**], the field equation component $(t\,\phi)$ to first order in the angular velocity writes [with the abbreviations (25) in that paper]

$$\begin{aligned} \frac{\partial}{\partial r}\left[r^4 j(r)\frac{\partial \omega}{\partial r}\right] + \frac{r^2\, k(r)}{\sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta\frac{\partial\omega}{\partial\theta}\right] &+ 4\,r^3\frac{dj}{dr}(r)\;\omega \\ &= 4\,r^3\frac{dj}{dr}(r)\;\Omega(r,\theta)\,. \end{aligned} \tag{1}$$

which can be seen as a differential equation for the dragging rate $\omega \equiv \omega(r,\theta)$, with the fluid angular velocity $\Omega \equiv \Omega(r,\theta)$ in the inhomogeneity term.

Remarkably, due to the assumed energy condition,

$$\frac{dj}{dr} = -4\pi\, r\,(\varepsilon_0 + p_0)\,k\, < 0 \quad \text{in } [0,R[\;\; (\text{interior})\,, \tag{2}$$

$dj/dr\; \equiv 0$ in $[R,\infty[$ (exterior), where vacuum ($\varepsilon_0 \equiv p_0 \equiv 0$) is considered [hence, Eq. (1) is homogeneous there]. Furthermore, $j \equiv 1$ in the exterior [cf. (25) and (20) in **Paper II**].

To solve Eq. (1) by separation of variables, we use an expansion in vector spherical harmonics. The associated homogeneous equation satisfied by an ω,

$$\frac{\partial}{\partial r}\left[r^4 j(r)\frac{\partial\omega}{\partial r}(r,\theta)\right] + \frac{r^2\,k(r)}{\sin^3\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta\frac{\partial\omega}{\partial\theta}(r,\theta)\right] + 4\,r^3\frac{dj}{dr}(r)\,\omega(r,\theta) = 0\,, \tag{3}$$

is separable by the ansatz $\omega(r,\theta) = R(r)\,Y(\theta)$, and yields

a *radial* equation, $\quad [r^4 j(r) R'(r)]' + [4\,r^3 j'(r) - r^2\,k(r)\,\lambda]\;R(r) = 0\,, \qquad (4)$

$$\text{and an } \textit{angular} \text{ equation,} \quad [\sin^3\theta\,\dot{Y}(\theta)]^{\cdot} + \lambda\sin^3\theta\,Y(\theta) = 0\,, \tag{5}$$

where $\lambda \in \mathbb{R}$ and the notation for the derivatives $d/dr \equiv {}'$ and $d/d\theta \equiv {}^{\cdot}$ has been adopted.

We first solve Eq. (5), and obtain that for each $\lambda \equiv \lambda_l = l(l+1) - 2, \quad l \in \mathbb{N},\ l \neq 0$ there is a regular (at least, class C^1) solution, namely, with the coordinate change $\theta \to x := \cos\theta$, and denoting $\mathcal{P}_l \equiv$ Legendre polynomial of degree l,

$$Y_l(\theta) \equiv y_l(x) := \frac{d\mathcal{P}_l}{dx} \quad \forall x \in [-1,1] \quad (\theta \in [0,\pi])\,. \tag{6}$$

Consequently, $\omega(r,\theta) \equiv \omega(r,x) = \sum_{l=1}^{\infty} R_l(r)\,y_l(x)$, with R_l solution of the radial equation (4) for $\lambda = \lambda_l$, and y_l given by[2] (6), is solution of the homogeneous equation (3).

The system $\{y_l\}_{l=1}^{\infty}$ is orthogonal in the Hilbert space[3]

$$L^2_\rho([-1,1]) \equiv \{f : [-1,1] \longrightarrow \mathbb{R},\ \text{measurable} \ / \int_{-1}^{1} \rho(x)[f(x)]^2 dx < +\infty\}$$

—with the scalar product $\langle f, g\rangle_\rho := \int_{-1}^{1} \rho(x) f(x) g(x) dx \quad \forall f, g \in L^2_\rho([-1,1])$, which defines the norm $\|f\|_\rho := (\langle f, f\rangle_\rho)^{1/2}$— with respect to the weight function $\rho(x) := 1 - x^2,\ x \in [-1,1]$, and the polynomials y_l have norm $\|y_l\|_\rho^{\,2} = 2l(l+1)/(2l+1)$. [These polynomials are indeed directly related to a class of the so-called Jacobi polynomials. See Appendix B, where convergence of these series is discussed.]

Let $\omega(r,\theta)$ be a solution of the complete equation (1) such that, for each $r \geq 0$ fixed, it can be expressed

$$\omega(r,\theta) \equiv \omega(r,x) \sim \sum_{l=1}^{\infty} \omega_l(r)\,y_l(x), \quad \text{where } y_l \text{ is given by (6)}\,.$$

We next try to expand our term of inhomogeneity in Eq. (1) as well such that, for each $r \geq 0$ fixed,

$$\Omega(r,\theta) \equiv \Omega(r,x) \sim \sum_{l=1}^{\infty} \Omega_l(r)\,y_l(x), \quad \text{where } y_l \text{ is given by (6) and}$$

$$\Omega_l(r) \equiv \frac{1}{\|y_l\|^2_\rho} \int_{-1}^{1} (1-x^2)\Omega(r,x)\,y_l(x) dx\,. \tag{7}$$

Then each ω_l should satisfy the radial equation

$$[r^4 j(r)\ \omega_l']' + [4\,r^3 j'(r) - r^2\,k(r)\,\lambda_l]\ \omega_l = 4\,r^3 j'(r)\ \Omega_l(r)\,. \tag{8}$$

Accordingly, the series expansion

$$\omega(r,\theta) \equiv \omega(r,x) \sim \sum_{l=1}^{\infty} \omega_l(r)\, y_l(x)\,, \tag{9}$$

where y_l is given by (6) and ω_l satisfies (8), is (formal) solution of the complete equation (1) in $r \in [0,\infty[,\ x \in [-1,1]$.

Specifically, we are only interested in solutions ω of Eq. (1) which, as functions of r, are

a. $\omega(.,x)$ asymptotically flat (at $r \to \infty$) and

b. $\omega(.,x)$ regular at the origin $r = 0$,
(*boundary* conditions)

apart from

c. $\omega(.,x)$ at least class C^1 on the surface of the star (spherical at first order rotational perturbations), that is, C^1 across $r = R$ (r, Gaussian coordinate with respect to the star's surface);
(a matching condition)

Hence, in particular, we shall consider only solutions ω_l of Eq. (8) in $[0,\infty[$ with the boundary (resp. matching) conditions

a'. ω_l asymptotically flat,

b'. ω_l regular at $r = 0$, and

c'. ω_l a class C^1 function across $r = R$.

Notice, condition (a) [and (a')] follows from our star model, (b) [and (b')] follows from the regularity of $\Omega(.,x)$ at $r = 0$ and the boundedness of $\omega(.,x)$ (cf. Appendix A), and (c) [and (c')] is already satisfied, provided that the dragging rate, ω, and the fluid angular velocity, Ω, as functions of r, are (at least) totally bounded almost everywhere —i.e. even if it has a jump discontinuity—, $\Omega(.,x) \in L^\infty([0,R])$ [cf. Appendix A, Property (a)].

Remarkably, since $j' < 0$ in $[0,R[$ [cf. Eq. (2)] and $j' \equiv 0$ in $[R,+\infty[$, the coefficient of ω_l in Eq. (8) is non-positive, i.e.

$$h_l(r) \equiv 4\, r^3 j'(r) - r^2\, k(r)\, \lambda_l \le 0 \qquad \forall r \ge 0\,. \tag{10}$$

The first observation is

Proposition 1

With the boundary conditions (a') and (b'), and the matching condition (c'),

$$\Omega_l \equiv 0 \;\Leftrightarrow\; \omega_l \equiv 0 \qquad \textit{for each}\;\; l \in \mathbb{N},\, l \neq 0\,.$$

Proof. From Eq. (8), obviously, $\omega_l \equiv 0 \Rightarrow \Omega_l \equiv 0$.

Suppose now $\Omega_l \equiv 0$.

In the INTERIOR,

integrating once Eq. (8), that is [with the notation in (10)],

$$\left[r^4 j(r)\; \omega_l'(r)\right]' + h_l(r)\; \omega_l(r) = 0 \qquad \forall\; 0 \leq r \leq R\,, \tag{11}$$

from the axis on, we obtain

$$r^4 j(r)\; \omega_l'(r) = -\int_0^r h_l(\bar{r})\; \omega_l(\bar{r})\; d\bar{r} \qquad \forall\; 0 \leq r \leq R\,, \quad \text{i.e.}$$

$$\omega_l'(r) = -\int_0^r \frac{h_l(\bar{r})}{r^4 j(r)} \omega_l(\bar{r})\; d\bar{r} \qquad \forall\; 0 \leq r \leq R\,. \tag{12}$$

On the other hand, the general solution of [the linear homogeneous second order ordinary differential] equation (11) will be

$$\omega_l(r) = {c_l}^1\; {g_l}^1(r) + {c_l}^2\; {g_l}^2(r)\,,$$

where ${g_l}^1$ and ${g_l}^2$ are the two fundamental solutions, which, by an ansatz of the form r^{S_l}, can be seen to have the following behavior on the axis[4,5]

$$\text{as } r \to 0, \quad {g_l}^1(r) \sim \frac{1}{r^{l+2}} \;\;\text{and}\;\; {g_l}^2(r) \sim r^{l-1}\,,$$

with constant coefficients ${c_l}^1$ and ${c_l}^2$, the first of which, by regularity at the origin [condition (b') above], vanishes, ${c_l}^1 = 0$, thus showing that the general solution of Eq. (11) has the form

$$\omega_l(r) = c_l\; g_l(r), \quad \text{with} \;\; g_l(r) \sim r^{l-1} \;\; \text{as} \;\; r \to 0 \;\; \text{and} \;\; c_l = \text{const.} \tag{13}$$

In the EXTERIOR,

from Eq. (8), taking into account $j(r) \equiv 1 \;\; \forall r > R$, one also has

$$\left[r^4\; \omega_l'(r)\right]' + h_l(r)\; \omega_l(r) = 0 \qquad \forall r > R, \quad \text{here} \;\; h_l(r) = -r^2\, k(r)\, \lambda_l\,, \tag{14}$$

which, integrated once from the star's surface on, yields

$$r^4\omega_l'(r) = R^4\omega_l'(R) - \int_R^r h_l(\bar r)\,\omega_l(\bar r)\,d\bar r \qquad \forall r > R\,, \quad \text{i.e.}$$

$$\omega_l'(r) = \left(\frac{R}{r}\right)^4 \omega_l'(R) - \int_R^r \frac{h_l(\bar r)}{r^4}\omega_l(\bar r)\,d\bar r, \quad \forall r > R\,. \tag{15}$$

Again, the general solution of Eq. (14) has the form

$$\omega_l(r) = b_l{}^1\ f_l{}^1(r) + b_l{}^2\ f_l{}^2(r)\,, \tag{16}$$

where $b_l{}^1$ and $b_l{}^2$ are integrating constants and $f_l{}^1$ and $f_l{}^2$ are the two fundamental solutions with the asymptotic behavior[6]

$$\text{as } r \to \infty, \quad f_l{}^1(r) \sim \frac{1}{r^{l+2}} \ \text{ and } \ f_l{}^2(r) \sim r^{l-1}\,;$$

from where, by asymptotic flatness [condition (a') above], $b_l{}^2 = 0$.

Let us now suppose $c_l \neq 0$ [in Eq. (13)], for instance, $c_l > 0$ [and assume, naturally, $g_l(r) > 0$ at least for small r], then $\omega_l > 0$ in $[0, R]$ and, by (12), $\omega_l' > 0$ in $[0, R]$; in particular, $\omega_l(R) > 0$ and $\omega_l'(R) > 0$, which, since, by continuity, $\omega_l(r) > 0$ for r in the neighborhood (on the right) of R, by formula (15) on the exterior and by the C^1-matching condition (c'), it follows $\omega_l' > 0$ also in $]R, \infty[$, and therefore $\omega_l' > 0$ everywhere, that is, the global solution ω_l is a strictly increasing function. Analogously, for the case $c_l < 0$, ω_l is a strictly decreasing function. But that strict monotonicity, starting from $\omega_l(r) \to 0$ as $r \to 0$ for $l \geq 2$ [$\omega_1(r) \to$ const. $\neq 0$, in general] is in contradiction to the asymptotic flatness. Hence, we have $c_l = 0$ (trivial interior solution) and in particular, $\omega_l(R) = \omega_l'(R) = 0$, which, by the C^1-matching, are initial conditions for the exterior problem, but these are satisfied by the trivial exterior solution ($b_l{}^1 = b_l{}^2 = 0$), and, by the uniqueness theorem of ordinary differential equations, this is the only solution, $\omega_l \equiv 0$. $\square$

Remark 1. Since the Legendre polynomial of degree 1 is $\mathcal{P}_1(x) \equiv x$, then $y_1 \equiv 1$, and the first term of the series expansions of Ω and ω is free of angular part,

$$\Omega(r,x) \sim \Omega_1(r) + \sum_{l=2}^{\infty} \Omega_l(r)\, y_l(x), \quad \omega(r,x) \sim \omega_1(r) + \sum_{l=2}^{\infty} \omega_l(r)\, y_l(x)\,.$$

Consequently, from the previous result (Proposition 1), it follows

$$\Omega(r,x) = \Omega(r) \equiv \Omega_1(r) \quad \Leftrightarrow \quad \omega(r,x) = \omega(r) \equiv \omega_1(r)\,.$$

Notice, that shows in particular, that in the case of **rigid rotation** [$\Omega(r,x) =$ const.], included in this particular differential rotation case, we have $\omega(r,x) = \omega(r)$, in accordance with the results in Ref. 4. We shall consider here the most general case of slow differential rotation, and we shall therefore suppose $\Omega_{l_0} \not\equiv 0$ for some $l_0 \geq 2$, i.e. $\partial\Omega/\partial x \not\equiv 0$.

II. QUALITATIVE BEHAVIOR OF THE RADIAL PART OF ω. COMPARISON WITH THE RADIAL PART OF Ω

An interior solution of Eq. (8), for each $l \in \mathbb{N}$ $l \neq 0$, can be written as

$$\omega_l(r) = {x_l}^p(r) + c_l\ g_l(r) \quad \forall\ 0 \leq r \leq R\,, \tag{17}$$

where ${x_l}^p$ is a particular solution, the one corresponding to a given Ω_l, and $c_l g_l$ is the general solution, g_l solution of the associated homogeneous equation, and, hence, by Eq. (12), a strictly monotonic function, with $g_l(r) \sim r^{l-1}$ as $r \to 0$, and $c_l =$ const.

The regularity of the particular solution ${x_l}^p$ is determined by the regularity of the right hand side of Eq. (8), term of inhomogeneity, specifically of Ω_l. In particular, if $\Omega_l \in C^0(]0,R[)$ and $\omega_l \equiv {x_l}^p$ is essentially bounded, $\omega_l \in L^\infty([0,\infty[)$, then $\omega_l \equiv {x_l}^p \in C^2(]0,\infty[\backslash\{R\}) \cap C^1(\mathcal{N}_R)$ [$\mathcal{N}_R$, a neighborhood of $r = R$] [cf. Appendix A, Property (a)]. Obviously, g_l is analytical, like the exterior solution, ${f_l}^1$, since both are solutions of an ordinary differential equation with analytic coefficient functions. We shall suppose $\Omega(.,x) \in C^{0,\gamma}$ at $r = 0$ for some $\gamma < 1$, leading to $\omega(.,x) \in C^{2,\gamma}$ at $r = 0$, cf. Appendix A, Property (b). [Notice, continuity of $\Omega(.,x)$ at $r = 0$, considering its series expansion, yields in particular $\Omega_l(0) = 0$ $\forall l \geq 2$, what holds if, for instance, its behavior at the origin is $\Omega_l(r) \sim r^{m_l}$, with $m_l \geq 1$, as $r \to 0$, $\forall l \geq 2$; then, assuming that ${x_l}^p$ (and its first and second derivatives) goes near $r = 0$ like some power of r, ${x_l}^p(r) \sim r^{P_l}$, for small r, it follows, via Eq. (8), $P_l = m_l + 2$, that is, ${x_l}^p(r) \sim r^{m_l+2}$, with $m_l \geq 1$, as $r \to 0$, $\forall l \geq 2$, from where $\omega_l(0) = 0$ $\forall l \geq 2$; but then Eq. (8) guarantees $\omega_l'(0) = 0$ $\forall l \geq 1$ as well, in consistency with $\omega(.,x)$ at least class C^2 at $r = 0$.]

Once Ω is given, for each $l \geq 1$ with $\Omega_l \not\equiv 0$ (given) a particular solution of Eq. (8), $\omega_l \equiv {x_l}^p \not\equiv 0$, is determined; g_l is a fundamental solution [in (17)], and, from the C^1-matching on the surface of the star $r = R$ [condition (c')], namely, $\omega_l(R)|_{int} = \omega_l(R)|_{ext}$ and $\omega_l'(R)|_{int} = \omega_l'(R)|_{ext}$, for each constant c_l, the coefficients of the exterior solution ${b_l}^1$ and ${b_l}^2$ are determined. Remarkably, we can uniquely choose c_l such that ${b_l}^2 = 0$, that is, such that the so considered interior solution ω_l, Eq. (17), satisfying regularity at the origin, (b'), matches, (c'), to an exterior solution ω_l asymptotically flat, (a'), i.e. with the fall off behavior $\omega_l(r) \sim {b_l}^1 r^{-l-2}$ as $r \to \infty$. Namely, from ${b_l}^2 = 0$ it follows

$$c_l = \left. \frac{-{f_l}^1 {x_l}^{p'} + {f_l}^{1'} {x_l}^{p}}{{f_l}^1 g_l' - {f_l}^{1'} g_l} \right|_{r=R} . \tag{18}$$

Note, the denominator of (18) does not vanish, which can be seen as follows: by Eq. (12) for $\omega_l \equiv g_l$, homogeneous interior solution of Eq. (8), assuming $g_l(r) > 0$ for small r [remember, $g_l(r) \sim r^{l-1}$ as $r \to 0$], it follows $g_l' > 0$ (g_l strictly increasing function), in particular $g_l'(R) > 0$. Also, by Eq. (15) for $\omega_l \equiv {f_l}^1$, exterior (homogeneous) solution of Eq. (8), [recall, ${f_l}^1(r) \sim r^{-l-2}$ as $r \to \infty$] take without restriction ${f_l}^1(r) > 0$ for large r, it follows ${f_l}^1(r) > 0$ for r in the neighborhood on the right of R as well, then ${f_l}^{1'}(R) < 0$ (and hence ${f_l}^{1'} < 0$ everywhere in $[R, \infty[$), for if ${f_l}^{1'}(R) \geq 0$, then ${f_l}^{1'}(r) > 0$ in particular for large r, contradicting the asymptotic flatness. We have then $g_l'(R) > 0$ and ${f_l}^{1'}(R) < 0$, hence the denominator of (18) does not vanish, and furthermore, in the quotient given by the C^1-matching at $r = R$,

$$\left. \frac{\omega_l'(R)}{\omega_l(R)} \right|_{int} = \left. \frac{\omega_l'(R)}{\omega_l(R)} \right|_{ext} = \frac{{f_l}^{1'}(R)}{{f_l}^1(R)} = \sigma \equiv \text{ const.}, \tag{19}$$

we therefore have $\sigma < 0$, and, from the relation

$$\omega_l'(R) = \sigma\, \omega_l(R) \qquad [\sigma < 0]\,, \tag{20}$$

if $\omega_l(R) > =$ or < 0 (which depends on the sign of Ω_l, see below), then $\omega_l'(R) < =$ or > 0, respectively.

Remarkably, from Eq. (8), ω_l will satisfy the following differential inequality [with the notation in (10)], provided that $\Omega_l \geq 0$,

$$\left[r^4 j(r)\, \omega_l'\right]' + h_l(r)\, \omega_l = 4\, r^3 j'(r)\, \Omega_l \leq 0\,,$$

that is, dividing by $r^4 j(r)$ (> 0), ω_l will satisfy, in particular in the interval $]0, R[$, where $\omega_l \in C^0(]0, R[)$, the inequality

$$\omega_l'' + b(r)\, \omega_l' + \tilde{h}_l(r)\, \omega_l \leq 0\,, \tag{21}$$

where it has been denoted

$$b(r) \equiv \frac{4}{r} + \frac{j'(r)}{j(r)}, \qquad \tilde{h}_l(r) \equiv \frac{h_l(r)}{r^4 j(r)} \equiv \frac{4}{r}\frac{j'(r)}{j(r)} - \frac{k(r)\ \lambda_l}{j(r)\, r^2}\,. \tag{22}$$

Both b and $\tilde{h}_l$ are bounded functions on every closed subinterval of $]0, R[$, and remarkably $\tilde{h}_l \leq 0$ (we had seen $h_l \leq 0$). Thus a (strong) *minimum* principle (see, e.g., Theorem 3 in Ref. 7, p. 6, for the function $-u$) applies to the differential inequality (21) and, as a result, ω_l cannot attain a non-positive minimum at an interior point of $[0, R]$, unless $\omega_l \equiv$ const. This and the matching at $r = R$ to a monotonic asymptotically flat exterior solution allows us to prove the following results. Let us first study the case $l \geq 2$ [recall, we have assumed $\Omega_{l_0} \not\equiv 0$ for some $l_0 \geq 2$].

Proposition 2 ($\forall l \geq 2$)
If $\Omega_l \geq 0$ *(in* $[0, R]$*),* $\quad \Omega_l \in C^0(]0, R[)$, $\quad \Omega_l \not\equiv 0$
then $\omega_l > 0$ *in* $]0, \infty[$. *In particular,* $\omega_l(R) > 0$ *and* $\omega_l'(R) < 0$
$[\,\omega_l(0) = \Omega_l(0) = 0 \quad \forall l \geq 2$, *from continuity at* $r = 0$ *of* $\omega(.,x)$ *and* $\Omega(.,x)$*;*[8] *hence,* $\omega_l \geq 0$ *in* $[0, \infty[\,]$.

Proof. Let us first prove $\omega_l(R) \geq 0$. Suppose $\omega_l(R) < 0$. Then, by Eq. (20), $\omega_l'(R) > 0$. But from $\omega_l(0) = 0$ and $\omega_l(R) < 0$ with $\omega_l'(R) > 0$ it follows that $\exists\, r_1 \in]0, R[$ such that $\omega_l(r_1) < \omega_l(R) < 0$ and $\omega_l(r_1) < \omega_l(0) = 0$, contradicting the minimum principle mentioned above. That shows that $\omega_l(R) \geq 0$, which, together with $\omega_l(0) = 0$, yields, via the same minimum principle, $\omega_l \geq 0$ in $[0, R]$, but also, by the matching condition, $\omega_l \geq 0$ everywhere in $[0, \infty[$. Furthermore, $\omega_l(R) \geq 0$ yields, via Eq. (20), $\omega_l'(R) \leq 0$. But the case $\omega_l(R) = \omega_l'(R) = 0$ is not possible for a non-trivial solution $\omega_l \not\equiv 0$, corresponding to an $\Omega_l \not\equiv 0$ (see Proposition 1), because it has been seen $\omega_l \geq 0$ in particular in $[0, R]$, and if $\omega_l(R) = 0$, that means $\omega_l(R)$ is a minimum, and again by the minimum principle applied to inequality (21) (see, e.g., Theorem 4 in Ref. 7, p. 7), it follows $\omega_l'(R) < 0$, which contradicts $\omega_l'(R) = 0$. Hence, $\omega_l(R) > 0$ [and $\omega_l'(R) < 0$], following, from that minimum principle, $\omega_l > 0$ in $]0, \infty[$ (see Corollary of Theorem 4 in Ref. 7, p. 7, for the function $-u$). $\square$

Remark 2. For $\Omega_l \geq 0$ (in $[0, R]$), it has been seen (Proposition 2) $\omega_l > 0$ in $]0, \infty[$. Observe, since $\omega_l(R) > 0$ and $\omega_l'(R) < 0$, having [from continuity of $\omega(.,x)$ on the axis] $\omega_l(0) = 0 \quad (l \geq 2)$, we essentially have that ω_l reaches an maximum at an interior point (at least one) and falls off as r^{-l-2} as $r \to \infty$. See Fig. 1. Notice, since $b(r)$ is not bounded below at $r = 0$, $0 = \omega_l'(0) = \omega_l(0) = \min \omega_l$ yields no contradiction.

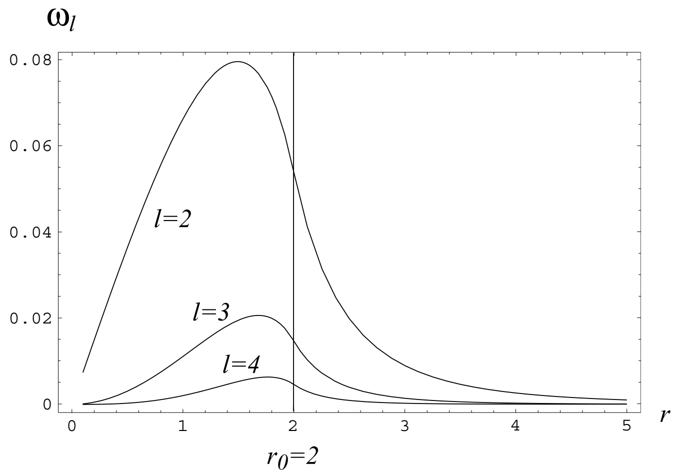

FIG. 1. ω_l for $l = 2,\ 3$ and 4, starting from a non-rotating configuration of incompressible ($\varepsilon_0 =$ const.) perfect fluid (interior Schwarzschild solution) with total mass $M = 0.5$ ($\varepsilon_0 \simeq 0.1492$) and radius $R = 2$, for $\Omega_l(r) = (r/4)^{l-1} \quad 0 \le r \le R$.

The function difference $\omega_l - \Omega_l$ can be studied with a similar reasoning, because from Eq. (8) for ω_l, divided by j (> 0), it follows

$$\omega_l'' + \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right)\omega_l' + \left(\frac{4}{r}\frac{j'(r)}{j(r)} - \frac{k(r)}{j(r)}\frac{\lambda_l}{r^2}\right)\omega_l = \frac{4}{r}\frac{j'(r)}{j(r)}\Omega_l\,, \tag{23}$$

which can be written —we shall assume that Ω_l'' exists almost everywhere and is locally bounded— in the form

$$\begin{aligned}(\omega_l - \Omega_l)'' + \left(\tfrac{4}{r} + \tfrac{j'(r)}{j(r)}\right)(\omega_l - \Omega_l)' + \left(\tfrac{4}{r}\tfrac{j'(r)}{j(r)} - \tfrac{k(r)}{j(r)}\tfrac{\lambda_l}{r^2}\right)(\omega_l - \Omega_l)\\ = -\Omega_l'' - \left(\tfrac{4}{r} + \tfrac{j'(r)}{j(r)}\right)\Omega_l' + \tfrac{k(r)}{j(r)}\tfrac{\lambda_l}{r^2}\Omega_l\,,\end{aligned}$$

so that, for Ω_l satisfying

$$-\Omega_l'' - \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right)\Omega_l' + \frac{k(r)}{j(r)}\frac{\lambda_l}{r^2}\Omega_l \ge 0\,, \tag{24}$$

$u_l \equiv \omega_l - \Omega_l$ will satisfy the differential inequality

$$u_l'' + \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right)u_l' + \left(\frac{4}{r}\frac{j'(r)}{j(r)} - \frac{k(r)}{j(r)}\frac{\lambda_l}{r^2}\right)u_l \ge 0\,,$$

i.e., using the notation in (22),

$$u_l'' + b(r)\ u_l' + \tilde{h}_l(r)\ u_l \ge 0\,. \tag{25}$$

As already remarked, the functions b and $\tilde{h}_l$ are bounded on every closed subinterval of $]0, R[$, and $\tilde{h}_l \leq 0$. This enables us to apply a (strong) maximum principle (see, e.g., Theorem 3 in Ref. 7, p. 6) to inequality (25), which is satisfied by u_l in particular in the interval $]0, R[$, where $u_l \in C^0(]0, R[)$ [assuming $\Omega_l \in C^0(]0, R[)$], and consequently, u_l cannot attain a non-negative maximum at an interior point of $[0, R]$, unless $u_l \equiv$ const., that is, $\omega_l \equiv \Omega_l +$ const., actually [since $\omega_l(0) = \Omega_l(0), \ \forall l \geq 2$] $\omega_l \equiv \Omega_l$; our requirements however will exclude this case. That gives the following result

Proposition 3 ($\forall l \geq 2$)
For $\Omega_l \not\equiv 0$ satisfying

α. $\Omega_l \in C^0(]0, R[)$, $\exists\, \Omega_l''$ *almost everywhere and is locally bounded,*

β. $\Omega_l \geq 0$ *(in $[0, R]$),*

γ.

$$-\Omega_l'' - \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right)\Omega_l' + \frac{k(r)}{j(r)}\frac{\lambda_l}{r^2}\Omega_l \geq 0 \quad (\textit{in } [0, R]) \,, \quad \textit{and}$$

δ. $\Omega_l'(R) \geq 0$,

$0 < \omega_l(r) < \Omega_l(r)$ *holds* $\forall r \in]0, R]$ $\quad$ [$\omega_l(0) = \Omega_l(0) = 0 \ \ \forall l \geq 2$, *from continuity at* $r = 0$].

Proof. First notice, by conditions (α) and (β), Proposition 2 applies, and we have in particular $\omega_l'(R) < 0$, which together with condition (δ), yields $u_l'(R) = \omega_l'(R) - \Omega_l'(R) < 0$. Condition ($\gamma$) is (24), hence $u_l \in C^0(]0, R[)$ satisfies inequality (25), to which, by (α), the above mentioned maximum principle applies. Let us first prove $u_l(R) < 0$. We have seen $u_l'(R) < 0$ and $u_l(0) = \omega_l(0) - \Omega_l(0) = 0$. Hence, if $u_l(R) \geq 0$, then the function u_l would reach a positive maximum at an interior point, contradicting the maximum principle. Therefore $u_l(R) < 0$, which, together with $u_l(0) = 0$, gives, via the same maximum principle, $u_l < 0$ in $]0, R]$, i.e. $0 < \omega_l < \Omega_l$ in $]0, R]$. □

Remark 3. Notice, the requirements in Proposition 3 yield also directly $\Omega_l(R) > 0$. In fact, by (α) and (γ), a maximum principle (for $-\Omega_l$) applies to the differential inequality (γ) of Proposition 3 in $[0, R]$. From (β) in particular $\Omega_l(R) \geq 0$, but if $\Omega_l(R) = 0$, then for $\Omega_l \geq 0$ [condition(β)] $\Omega_l(R)$ would be a minimum, and $\Omega_l'(R) < 0$, but this contradicts condition (δ). It follows then $\Omega_l(R) > 0$. See Fig. 2.

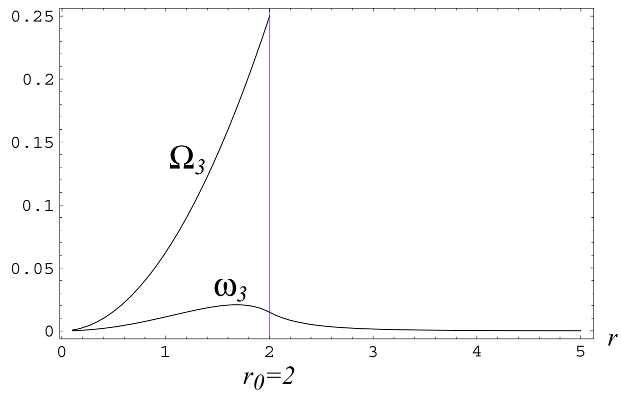

FIG. 2. $\Omega_3(r) = (r/4)^2 \quad 0 \leq r \leq R$ and the corresponding solution ω_3, starting from a non-rotating configuration of incompressible perfect fluid with total mass $M = 0.5$ and radius $R = 2$ ($l = 3$ in Fig. 1.)

For $\boldsymbol{l}$ $= 1$, however, $\omega_1(0) \neq 0 \;\; (\Omega_1(0) \neq 0)$ yields no contradiction to the continuity of $\omega(., x)$ [of $\Omega(., x)$], because, as already stressed, the Legendre polynomial of degree 1 is $\mathcal{P}_1(x) = x$, and thus $y_1 \equiv \frac{d\mathcal{P}_1}{dx} \equiv 1$, so that the first term in the series expansion (9) of $\omega(r, x)$ [of $\Omega(r, x)$] (for each r fixed) does not depend on the angular coordinate x (cf. Remark 1 above), in contrast to the $l \geq 2$ case, where we have the x-dependence.

We shall consider in the following results $\Omega_1 \geq 0,\; \Omega_1 \not\equiv 0,\; \Omega_1 \in C^0(]0, R[)$. Let us suppose, without loss of generality, that $0 < \Omega_1(0) < \infty$, or $\Omega_1(r) \sim$ const. > 0 [$\Omega_1(r) = O(1)$] as $r \to 0$, then, just from the boundedness of ω_1 and Ω_1 it follows [cf. Appendix A, Property (a)] $\omega_1'(r) = O(r)$ as $r \to 0$, specially, $\omega_1'(0) = 0$. In particular, assuming that $\omega_1 \equiv {x_1}^p$ (and its first and second derivatives) goes near $r = 0$ like some power of r, by Eq. (8), it follows ${x_1}^p(r) \sim r^2$ as $r \to 0$. On the other hand, g_1, an homogeneous interior solution of that equation for $l = 1$, behaves $g_1(r) \sim$ const. as $r \to 0$, therefore near the star's center, (17) for $l = 1$,

$$\omega_1(r) = {x_1}^p(r) + c_1 \; g_1(r) \quad \sim \quad \text{const.}' r^2 + c_1 \; \text{const.} \qquad \text{as } r \to 0\,,$$

and the dominant term for small r in ω_1 is $c_1 g_1$. The homogeneous solution, g_1, satisfies Eq. (11) for $l = 1$ (notice $\lambda_1 = 0$), and therefore

$$g_1'(r) = -4 \int_0^r \frac{\bar{r}^3}{r^4} \frac{j'(\bar{r})}{j(r)} g_1(\bar{r}) d\bar{r} \,,$$

from where, taking (without restriction) $g_1(0) > 0$, it follows $g_1' > 0$ (g_1 strictly increasing), and $g_1 > 0$. ω_1 will satisfy Eq. (8) for $l = 1$, with $\Omega_1 \not\equiv 0$ in the interior, i.e.

$$\left[r^4 j(r)\, \omega_1'(r)\right]' + 4\, r^3 j'(r)\, \omega_1(r) = 4\, r^3 j'(r)\, \Omega_1(r) \qquad \forall\, 0 \leq r \leq R \,, \tag{26}$$

or in its integral form

$$\omega_1'(r) = -4 \int_0^r \frac{j'(\bar{r})}{j(r)} \frac{\bar{r}^3}{r^4} \left[\omega_1(\bar{r}) - \Omega_1(\bar{r})\right] d\bar{r} \qquad \forall\, 0 \leq r \leq R \,. \tag{27}$$

Let us recall, this interior solution with the appropriate uniquely chosen constant c_1 matches to an asymptotically flat exterior (homogeneous, and therefore strictly monotonic) solution

$$\omega_1(r) = {b_1}^1 {f_1}^1(r) \quad \sim \quad {b_1}^1 \frac{1}{r^3} \;\; \text{where} \;\; {b_1}^1 = \;\text{const.} \;\; \text{as}\; r \to \infty \,,$$

satisfying

$$\omega_1'(r) = \left(\frac{R}{r}\right)^4 \omega_1'(R) \quad \forall r > R \,, \tag{28}$$

i.e. Eq. (15) for $l = 1$, where we have $\lambda_1 = 0$, and thus $h_1 \equiv 0$. Actually the exterior equation, Eq. (14), for $l = 1$ can be integrated exactly; that is, we integrate Eq. (28) once more, and, taking into account the condition of asymptotic flatness (${b_1}^2 = 0$), we get

$$\omega_1(r) = \left(\frac{R}{r}\right)^3 \omega_1(R) \equiv {b_1}^1 {f_1}^1(r) \quad \forall r > R \,, \tag{29}$$

take, e.g., ${f_1}^1(r) \equiv r^{-3}$, ${b_1}^1 \equiv R^3 \omega_1(R)$.

Proposition 4 ($l = 1$)
If $\Omega_1 \geq 0$ *(in* $[0, R]$), $\Omega_1 \in C^0(]0, R[)$, $\Omega_1 \not\equiv 0$
then $\omega_1 \geq 0$ *in* $[0, \infty[$,
$\omega_1 > 0$ *in* $]0, \infty[$. *In particular,* $\omega_1(R) > 0$ *and* $\omega_1'(R) < 0$.

Proof. For $\Omega_1 \geq 0$, ω_1 will satisfy inequality (21) for $l = 1$ [where $\tilde{h}_1(r) \leq 0$], to which (as already remarked above Proposition 2) a (strong) *minimum* principle applies. Let us first see $\omega_1(R) \geq 0$. Suppose $\omega_1(R) < 0$, then, by Eq. (20), $\omega_1'(R) > 0$. If $\omega_1(0) \geq \omega_1(R)$, then ω_1 would attain an non-positive minimum at an interior point (at least one), contradicting the mentioned minimum principle. In the other case, $\omega_1(0) < \omega_1(R)$ (< 0), it follows $\omega_1(r) < 0$ $\forall r \in [0, R]$ —for if $\exists\, r_1 \in\,]0, R[$ such that $\omega_1(r_1) \geq 0$, then ω_1 would reach a negative minimum at an interior point, again in contradiction to the minimum principle—, but $\omega_1 < 0$ in $[0, R]$ leads, via Eq. (27), to $\omega_1' \leq 0$ in $[0, R]$, which is not possible for $\omega_1(0) < \omega_1(R) < 0$. Therefore $\omega_1(R) \geq 0$. This yields directly, by Eq. (29), $\omega_1 \geq 0$ in the exterior, $\omega_1(r) = (R/r)^3\omega_1(R) \geq 0$ $\forall r > R$. But also $\omega_1(r) \geq 0$ $\forall r \in [0, R[$, because if $\exists\, \bar{r} \in [0, R[$ such that $\omega_1(\bar{r}) < 0$ $[\bar{r} \neq R$, since $\omega_1(R) \geq 0]$, then $\exists\, r_* \in [0, R[$ such that $\omega_1(r_*) = \min\, \omega_1 < 0$, and the only possibility which the minimum principle would allow is $r_* = 0$ so that $\omega_1(0) < 0$ and, by continuity, $\omega_1(r) < 0$ for small r; but, since $\Omega_1 \geq 0$, Eq. (27) would yield $\omega_1'(r) < 0$ for small r, ω_1 decreasing and negative function for small r, which with $\omega_1(R) \geq 0$, would reach a negative minimum at an interior point, contradicting again the minimum principle. Thus, also $\omega_1 \geq 0$ in $[0, R]$. But if $\omega_1(R) = 0$ $[\omega_1'(R) = 0$, by Eq. (20)], then $\omega_1(R)$ would be a minimum of ω_1 (≥ 0) in particular in $[0, R]$, following (see, e.g., Theorem 4 in Ref. 7, p. 7) $\omega_1'(R) < 0$, in contradiction to $\omega_1'(R) = 0$. Therefore $\omega_1(R) > 0$, and, by Eq. (29), $\omega_1 > 0$ in $[R, \infty[$. Also in the interior, since $\omega_1(0) \geq 0$ and $\omega_1(R) > 0$, the minimum principle (see, e.g., Corollary of Theorem 4 in Ref. 7, p. 7) yields $\omega_1 > 0$ in $]0, R]$. Hence, $\omega_1 > 0$ in $]0, \infty[$. □

Proposition 5 $(l = 1)$
If $\Omega_1(r) \equiv$ const. $\equiv \hat{\Omega}_1 > 0$,
then $0 < \omega_1(r) < \hat{\Omega}_1$ *in* $[0, R]$ *and* $\omega_1 > 0$ *in* $[0, \infty[$, $\omega_1' < 0$ *in* $]0, \infty[$, $\omega_1'(0) = 0$.

Proof. Notice, in this case, $\hat{\Omega}_1$ is a particular solution of Eq. (26), so that the interior solution can be written

$$\omega_1(r) = \hat{\Omega}_1 + c_1 g_1(r), \quad \text{with} \quad g_1(r) \sim \text{const.} \equiv 1 \quad \text{as} \quad r \to 0, \quad \text{and} \quad c_1 = \text{const.}$$

Remember, from the matching and *boundary* conditions c_1 was given by (18) for $l = 1$, yielding in the present case with ${x_1}^p(r) \equiv \hat{\Omega}_1 \equiv$ const. > 0 [and assuming, without loss of generality, $g_1 > 0$, ${f_1}^1 > 0$ so that $g_1' > 0$, ${f_1}^{1'} < 0$; see above at the beginning of Sec. II], it follows $c_1 < 0$. The first observation is that, since $c_1 g_1(r) < 0$, $\omega_1(r) < \hat{\Omega}_1$. Furthermore, $\omega_1'(r) = c_1 g_1'(r) < 0$ $\forall r \in]0, R]$; from where ω_1 is a strictly decreasing function in $]0, R]$. [This can be seen also directly from Eq. (27).] But, using Proposition 4, $\omega_1 > 0$ in $]0, R]$, and, hence, in $[0, R]$. Also, by the matching at $r = R$ with $\omega_1'(R) < 0$ $[\omega_1(R) > 0]$, Eq. (28) yields $\omega_1' < 0$ in $[R, \infty[$. Thus, $\omega_1 > 0$ everywhere in $[0, \infty[$, and $\omega_1' < 0$ in $]0, \infty[$ $[\omega_1'(0) = 0]$, falling off for large r as r^{-3}, Eq. (29). See Fig. 3. $\square$

Remark 5. Actually, in the particular case of *rigid rotation* $[\Omega(r,\theta) = \text{const.} \equiv \hat{\Omega}]$, $\omega_l \equiv 0$, $\forall l \geq 2$ [see Remark 1], and $\omega(r,\theta) = \omega(r) \equiv \omega_1(r) = \hat{\Omega} + c_1 g_1(r)$, with $g_1(r) \sim \text{const.} \equiv 1$ as $r \to 0$, and $c_1 = \text{const.}$, solution of Eq. (8) for $l = 1$. With the *boundary* and matching conditions, a non-trivial solution ω_1 can be seen to be strictly monotonic —namely, (assuming $g_1 > 0$) if $\hat{\Omega} > 0$ $(c_1 < 0)$, then $0 < \omega(r) \equiv \omega_1(r) < \hat{\Omega}$, and ω is strictly decreasing, and if $\hat{\Omega} < 0$ $(c_1 > 0)$, then $\hat{\Omega} < \omega(r) \equiv \omega_1(r) < 0$, and ω is strictly increasing—, with the fall off behavior $\omega(r) \equiv \omega_1(r) \sim {b_1}^1 r^{-3}$ as $r \to \infty$. The solution has the form $\omega_1(r) = 2Jr^{-3}$ $\forall r > R$ [e.g., ${f_1}^1(r) := r^{-3}$, ${b_1}^1 := 2J$], where the constant J can be identified with the total angular momentum of the star. The main observation is $|\omega(r)| < |\hat{\Omega}|$, the function $|\omega(r)|$ (dragging rate) taking its largest value at the center of the star. In accordance with results in Ref. 4.

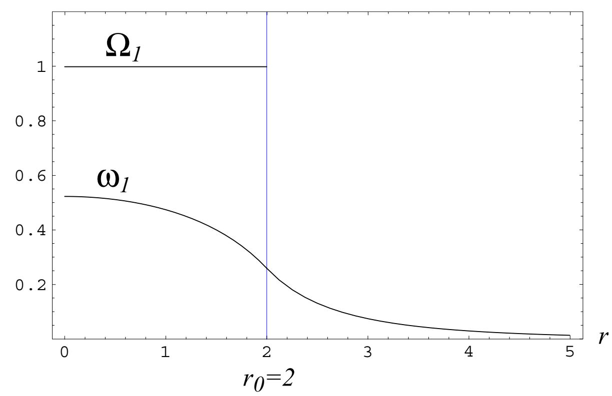

FIG. 3. ω_1 for $\Omega_1 = \text{const.} = 1$ $[= (r/4)^{l-1}$ for $l = 1]$, starting from a non-rotating configuration of incompressible perfect fluid with total mass $M = 0.5$ and radius $R = 2$.

III. COMPARING SUMS OF SERIES. POSITIVENESS OF $\Omega - \omega$

In view of the results obtained in Sec. II, particularly Propositons 3 and 5, one hopes to extend them, making use of some results on series expansions, and conlclude some positivenes result of the sum of the series difference, $\Omega - \omega$. In our attept to do it, we even conjecture the result below.

Notice, assuming $\Omega(.,x) \in C^0([0,R[)$ and furthermore class $C^{0,\gamma}$ (for some $\gamma < 1$)

at $r=0$, it follows in particular (cf. Appendix A)

$$\omega(.,x) \in C^2([0,\infty[\backslash\{R\}) \cap C^1(\mathcal{N}_R) \quad [\mathcal{N}_R, \text{ a neighborhood of } r=R]\,. \tag{30}$$

We have assumed $\omega(.,x)$ asymptotically flat, moreover, [by condition (viii) in Sec. II A of **Paper II**] bounded almost everywhere, $\omega(.,x) \in L^\infty([0,\infty[)$. Hence, all of conditions (a), (b), and (c) in Sec. I are satisfied. As a function of the angular coordinate x, ω is assumed

$$\omega(r,.) \in L^2_\rho([-1,1]) \cap C^1([-1,1]), \quad \rho(x) := 1-x^2\,. \tag{31}$$

With the assumptions made (for this slowly rotating configuration), if furthermore the distribution of angular velocity of the fluid, $\Omega \equiv \Omega(r,x)$ $[x \equiv \cos\theta]$, *satisfies*

I. $\Omega(.,x) \in C^0([0,R[)$, *and class* $C^{0,\gamma}$ *(for some* $\gamma < 1$*) at* $r=0$,
$\exists\, \frac{\partial^2\Omega}{\partial r^2}(.,x)$ *almost everywhere and is locally bounded,*

II. $\Omega(r,.) \in L^2_\rho([-1,1]) \cap C^1([-1,1]) \qquad [\,\rho(x) := 1-x^2\,]$,
$\exists\, \frac{\partial^2\Omega}{\partial x^2}$ *and is bounded with respect to* x *in* $[-1,1]$,

III. $\int_{-1}^{1} \rho(x)\frac{\partial\Omega}{\partial r}(r,x)dx \equiv 0$,

IV. $\Omega(0,x) > 0$,

V. $\frac{\partial\Omega}{\partial x} \not\equiv 0$, *and*

VI. *all even order* (≥ 2) *derivatives (in* x*) of the functions*

$$\Omega(r,.), \quad \frac{\partial\Omega}{\partial r}(R,.) \quad \textit{and} \quad -\frac{\partial^2\Omega}{\partial r^2}(r,.) - \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right)\frac{\partial\Omega}{\partial r}(r,.) + \frac{10}{r^2}\frac{k(r)}{j(r)}\Omega(r,.)$$

are non-negative,

then $\omega(r,x) \leq \Omega(r,x)$ $\forall r \in [0,R]$ $\forall x \in [-1,1]$, *and, consequently, the angular momentum density (to first order in the fluid angular velocity) is* ≥ 0, *vanishing on the axis.*

Remark 1.1. If $\exists\, \frac{\partial^2\Omega}{\partial x^2}$ bounded in $[-1,1]$ [in (II)], then in particular $\frac{\partial\Omega}{\partial x} \in \mathrm{Lip}_{[-1,1]}\delta$ $\forall\delta \leq 1$, i.e. it satisfies $\forall x,\bar{x} \in [-1,1]$ $\left|\frac{\partial\Omega}{\partial x}(x) - \frac{\partial\Omega}{\partial x}(\bar{x})\right| \leq K\,|x-\bar{x}|^\delta$ $\forall\delta \leq 1$ and for some constant K. Indeed, instead of this condition [in (II)], it suffices to assume $\frac{\partial\Omega}{\partial x} \in \mathrm{Lip}_{[-1,1]}\delta$, with $\delta > 1/2$.

Remark 1.2. Requirement (VI) is satisfied, for instance, if the even order (≥ 2) derivatives (in x) of these functions vanish up to a certain order, and the non-null ones are positive (see example in Sec. IV). Furthermore, the unperturbed non-rotating configuration satisfies $j'(r)/j(r) \leq 0$, and if the coefficient $\lambda(r)$ is an increasing function (what happens for instance for an incompressible perfect fluid), then $k(r)/j(r) = \exp[\lambda(r)]$ is also increasing, starting from $k(0)/j(0) = 1$, yielding $k(r)/j(r) \geq 1$. In such a case it will be sufficient to consider

$$-\frac{\partial^2\Omega}{\partial r^2}(r,.) - \frac{4}{r}\frac{\partial\Omega}{\partial r}(r,.) + \frac{10}{r^2}\Omega(r,.)$$

instead of the last function in (VI) for these inequalities.

Remark 1.3. Notice, we are not necessarily restricted to the case of uniformly positive rotation, $\Omega(r,x) \geq 0$. However, requirements (III) and (IV) say that we are *perturbing around a* first Fourier-Jacobi coefficient $\Omega_1(r) \equiv \text{const.} > 0$

Using some results on Jacobi-Fourier series, which are actually the series expansions we have, (cf. Appendixes B and C) the attepts to proof the former conjecture have not yet been successfull. Instead, we see the following example, where the rotation profile of a slowly rotating configuration is constructed so that the angular momentum density is positive.

IV. AN EXAMPLE

Consider, as fluid angular velocity of the configuration, a function of the form

$$\Omega(r,x) = a(r) + c(r)x^2\,, \tag{32}$$

where the radial functions a and c are in $C^0([0,R[)$, such that $\exists\, a''$ and c'' almost everywhere and are locally bounded [condition (I) of Sec. III]. $c \not\equiv 0$ (V) will have to satisfy furthermore $c(0) = c'(0) = 0$. Obviously, (32) as a function of x, $\Omega(r,.)$, satisfies (II). The determinant conditions will be however the differential inequalities of (VI), which for (32) reduce to

$$c(r) \geq 0 \tag{33}$$

$$c'(R) \geq 0 \tag{34}$$

$$c''(r) + \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right) c'(r) - \frac{10}{r^2}\frac{k(r)}{j(r)} c(r) \leq 0\,. \tag{35}$$

But we can always choose c as the solution of the homogeneous equation

$$c'' + \frac{4}{r}c' - \frac{10}{r^2}c = 0\,,$$

non-divergent at $r = 0$, that is, $c(r) = C_1 r^2$, with $C_1 \equiv$ const., where the multiplying constant must be positive for (33) and (34) to be also satisfied. Notice, since the non-rotating configuration satisfies $j'/j \leq 0$ and [for $\lambda(r)$ increasing function, what happens in particular for incompressible perfect fluid] $k/j \geq 1$, it follows

$$c''(r) + \left(\frac{4}{r} + \frac{j'(r)}{j(r)}\right) c'(r) - \frac{10}{r^2}\frac{k(r)}{j(r)} c(r) \leq c''(r) + \frac{4}{r}c'(r) - \frac{10}{r^2}c(r) = 0\,,$$

and (35) holds. We then have

$$c(r) = C_1 r^2, \quad \text{with } C_1 > 0\,. \tag{36}$$

Conditions (III) and (IV) will now determine the function a in (32). In this case

$$\int_{-1}^{1} \rho(x)\frac{\partial \Omega}{\partial r}(r,x)dx = \frac{4}{3}[a'(r) + \frac{1}{5}c'(r)]\,,$$

hence condition (III) yields $a'(r) + 1/5\, c'(r) = 0$, that is,

$$a(r) = -\frac{1}{5}c(r) + C_2, \quad \text{with } C_2 \equiv \text{const.}$$

But, since $c(0) = 0$, $\Omega(0,x) = a(0) = C_2$, and (IV) gives $C_2 > 0$. We then have

$$a(r) = -C_1\frac{1}{5}r^2 + C_2, \quad \text{with } C_1, C_2 > 0\,, \tag{37}$$

and from (32), (36) and (37), it follows

$$\Omega(r,x) = C_2 - \frac{C_1}{5}r^2 + C_1 r^2 x^2, \quad \text{with } C_1, C_2 > 0\,. \tag{38}$$

Note, if the constants satisfy $C_2/C_1 \geq R^2/5$, then also $a(r) \geq 0$ $[0 \leq r \leq R]$ and, since $c(r) \geq 0$, we have $\Omega(r,x) \geq 0$.

We consider a non-rotating (unperturbed) configuration of incompressible perfect fluid with total mass $M = 0.5$ and radius $R = 2$ [note, $R/(2M) = 2 > 8/9$], to which we shall give a *small* angular velocity in the sense of Sec. IV in **Paper II**. The critical value of angular velocity —which would deform the star so greatly, becoming close to the *shedding mass* at its equator— is in this case $\Omega_{crit} \equiv (M/R^3)^{1/2} = 0.25$. Hence, we must choose the constants C_1 and C_2 in (38) such that

$$\max_{(r,x)\in[0,R]\times[-1,1]}|\Omega(r,x)| \ll \Omega_{crit} = 0.25\,.$$

Provided that C_1 and C_2 are also such that $C_2/C_1 \geq R^2/5 = 4/5$ [which is satisfied in particular if $C_2 \geq C_1$], $\Omega(r,x) > 0$ follows, and thus,

$$\begin{aligned}\max_{(r,x)\in[0,R]\times[-1,1]}\Omega(r,x) &= \max_{r\in[0,R]}\Omega(r,\pm 1) = \max_{r\in[0,R]}(C_2 + \frac{4}{5}C_1 r^2) = C_2 + \frac{4}{5}C_1 R^2 \\ &= C_2 + \frac{16}{5}C_1\,.\end{aligned}$$

We choose then the constants C_1 and C_2 such that

$$C_2 + \frac{16}{5}C_1 \ll 0.25 \quad \text{with} \quad C_2 \geq C_1\,.$$

Hence, taking for instance $C_1 = 1/125$ and $C_2 = 1/100$, the star would rotate *slowly* (in the sense of Sec. IV in **Paper II**) with (positive) angular velocity

$$\Omega(r,x) = \frac{1}{100} - \frac{1}{625}r^2 + \frac{1}{125}r^2x^2, \quad r \in [0, R = 2], \quad x \in [-1,1]\,. \tag{39}$$

In order to get (numerically) the corresponding dragging velocity function $\omega(r,x)$, solution of Eq. (1), we first notice that (39) can be written as series expansion in the orthogonal system of the derivatives of Legendre polynomials $\{y_l\}_{l=1}^{\infty} \equiv \{\frac{d\mathcal{P}_l}{dx}\}_{l=1}^{\infty}$, in this case with only two terms, namely,

$$\Omega(r,x) = \Omega_1(r) + \Omega_3(r)y_3(x), \quad r \in [0, R = 2], \quad x \in [-1,1]\,,$$

$$\text{where} \quad y_3(x) \equiv \frac{d\mathcal{P}_3}{dx}(x) = \frac{3}{2}(5x^2 - 1), \quad \text{remember} \quad y_1(x) \equiv 1,$$

with

$$\Omega_1(r) := \text{const.} = C_2 = \frac{1}{100} \quad \text{and} \quad \Omega_3(r) := \frac{2}{15}C_1 r^2 = \frac{2}{1875}r^2\,.$$

The corresponding dragging rate function (in the interior of the fluid)

$$\omega(r,x) = \omega_1(r) + \omega_3(r)y_3(x), \quad r \in [0, R = 2], \quad x \in [-1,1]\,,$$

with ω_1 and ω_3 the (numerical) solutions of the *radial* equation (8) for these Ω_1 and Ω_3, respectively, in the inhomogeneity term, satisfies (as expected, cf. Sec. III)

$$\omega(r,x) < \Omega(r,x), \quad r \in [0, R=2], \quad x \in [-1,1],$$

These functions are plotted in Figs. 4.

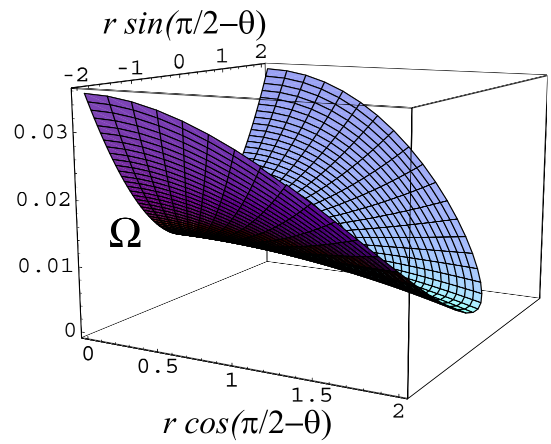

FIG. 4.1. $\Omega(r,\theta) = 1/100 - 1/625\,r^2 + 1/125\,r^2\cos^2\theta$, $\;r \in [0, R=2]$ and $\theta \in [0,\pi]$, given by (39) ($x \equiv \cos\theta$). [Note, each value of Ω and ω is plotted over a half-sphere, "euclidean" picture of the coordinate-range, representing a meridional cut across the rotating fluid ball. Remember, the geometry in this slowly rotating approximation, i.e. to first order in the angular velocity, is still the geometry of the static (global) solution.]

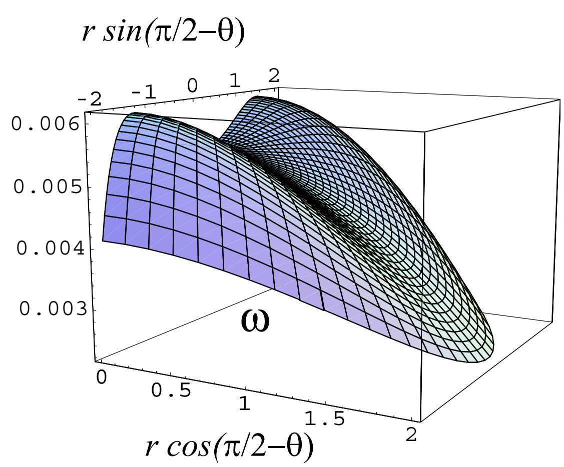

FIG. 4.2. Angular velocity of cumulative dragging (to first order in the angular velocity), ω, corresponding to a perturbed configuration whose angular velocity distribution is the one in Fig. 4.1, starting from a non-rotating spherical configuration of incompressible perfect fluid (interior Schwarzschild solution) with total mass $M = 0.5$ and radius $R = 2$.

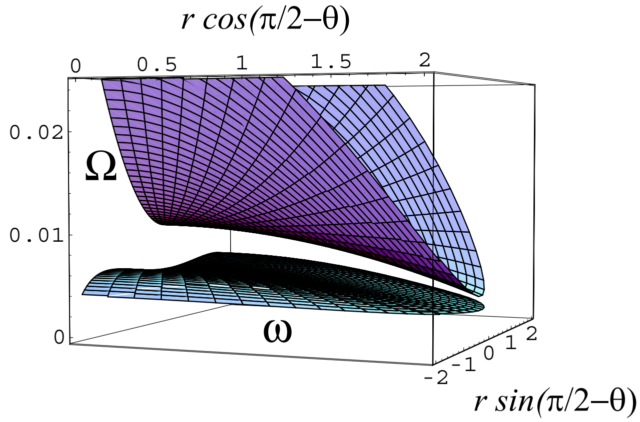

FIG. 4.3. Ω and ω plotted together.

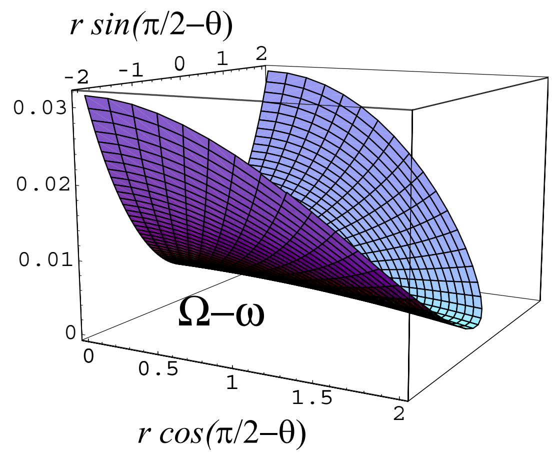

FIG. 4.4. Function difference $\Omega - \omega$.

APPENDIX A: Regularity

Let us consider a function ω_l satisfying

$$\left[r^4 j(r)\,\omega_l'(r)\right]' + [4\,r^3 j'(r) - r^2\,k(r)\lambda_l]\omega_l(r) = 4\,r^3 j'(r)\,\Omega_l(r) \quad \forall r \in [0, \infty[\,, \qquad \text{(A1)}$$

where $\lambda_l \equiv l(l+1) - 2$, $l \in \mathbb{N}$, $l \neq 0$, $j(r) \equiv \exp[-1/2\,(\lambda(r) + \nu(r))\,]$ and $k(r) \equiv \exp[1/2\,(\lambda(r) - \nu(r))\,]$, λ and ν coefficients of the starting non-rotating spherical metric described in **Paper II** (Remember $j' < 0$ in $[0, R[$, provided that the energy condition $\epsilon_0 + p_0 \geq 0$ holds, and $j' \equiv 0$ in $[R, \infty[$). The first result is

Property (a):

$$\begin{aligned} if \quad & \Omega_l \in C^0(]0,R[) \quad (\omega_l \in L^\infty([0,\infty[)), \\ then \quad & \omega_l \in C^2(]0,\infty[\backslash\{R\}) \cap C^1(\mathcal{N}_R) \;\; [\mathcal{N}_R, \; \textit{a neighborhood of } r = R]. \end{aligned} \tag{A2}$$

We first write Eq. (A1) in integral form,

$$\omega_l'(r) = \frac{\int_0^r \left[4\bar{r}^3 j'(\bar{r})\Omega_l(\bar{r}) - \left(4\bar{r}^3 j'(\bar{r}) - \bar{r}^2 k(\bar{r})\lambda_l\right)\omega_l(\bar{r})\right] d\bar{r}}{r^4 j(r)} \quad \forall r \in [0,\infty[\,, \tag{A3}$$

or, integrating once more, we could write ω_l as solution of a fixed-point equation. Consider first a generalized solution of that equation, satisfying the right *boundary* conditions at $r \to \infty$ and at $r = 0$, and essentially bounded,

$$\omega_l \in L^\infty([0,\infty[)\,. \tag{A4}$$

Let us denote the integrand in (A3) by

$$I_l(\bar{r}) \equiv 4\bar{r}^3 j'(\bar{r})\Omega_l(\bar{r}) - [4\bar{r}^3 j'(\bar{r}) - \bar{r}^2 k(\bar{r})\lambda_l]\omega_l(\bar{r})\,, \tag{A5}$$

so that (A3) now writes

$$\omega_l'(r) = \frac{\int_0^r I_l(\bar{r})d\bar{r}}{r^4 j(r)} \quad \forall r \in [0,\infty[\,. \tag{A6}$$

We have assumed (A4). If furthermore

$$\Omega_l \in L^\infty([0,R]) \tag{A7}$$

[which follows in particular for $\Omega_l \in C^0(]0,R[)$], then (A5) is a bounded function almost everywhere, $I_l \in L^\infty([0,\infty[)$, ($j$ and k are continuous functions). Therefore the map $r \mapsto h(r) := \int_0^r I_l(\bar{r})d\bar{r}$ is continuous, even Lipschitz-continuous. Hence, the numerator on the right hand side of (A6) is a Lipschitz-continuous function in $[0,\infty[$. The function j in the denominator of (A6) is also Lipschitz-continuous [because its derivative, j', has only a jump discontinuity at the star's surface $r = R$; outside and inside it is even C^∞]. Hence, the right hand side of (A6), and, therefore, ω_l' ($\forall l \geq 1$), is at least a class $C^{0,1}$ function everywhere except at $r = 0$ for $l \geq 2$ [due to the factor r^4 in the denominator, and the term $\bar{r}^2 k(\bar{r}) = O(\bar{r}^2), \;\; k(\bar{r}) = O(1)$, as $r \to 0$ in the numerator of (A6) if $l \geq 2$]. But this implies that ω_l ($\forall l \geq 1$) is at least a class $C^{1,1}$ function everywhere except at $r = 0$ for $l \geq 2$.

We shall study the regularity at $r = 0$ later on. So far, in the case $l = 1$, where $\lambda_{l=1}$ vanishes, and hence also the term with $\bar{r}^2 k(\bar{r})$, there is no problem at $r = 0$, because in this case the integrand, (A5), can be written in the form $\bar{r}^4$ multiplied by a bounded function, since $j'(\bar{r}) = O(\bar{r})$ as $\bar{r} \to 0$. Namely, since $j'(\bar{r}) \leq c_1 \bar{r}$ for some constant c_1, then, from (A5) for $l = 1$,

$$|I_1(\bar{r})| \leq 4c_1 \bar{r}^4 \left(\Omega_1(\bar{r}) - \omega_1(\bar{r})\right) = \bar{r}^4 \times [\text{a bounded function}] \quad \text{as} \quad r \to 0\,,$$

so that, $\left|\int_0^r I_1(\bar{r}) d\bar{r}\right| \leq \text{const.}\, \bar{r}^5$, and, therefore, since $j(r) \geq 1$, it follows from (A6) $|\omega_1'(r)| \leq \text{const.}'\, \bar{r}$, i.e. $\omega_1'(r) = O(\bar{r})$, as $r \to 0$, in particular, $\omega_1' \in C^{0,1}$ at $r = 0$ [specially it follows $\omega_1'(0) = 0$], and thus $\omega_1 \in C^{1,1}$ at $r = 0$. Hence, ω_1 is at least a class $C^{1,1}$ function everywhere in $[0, \infty[$.

We had already seen that, in particular, $\omega_l \in C^{1,1}(]0, \infty[)$ $(\forall l \geq 1)$. At the star's surface $r = R$ higher regularity of ω_l is not possible, due to the jump discontinuity of the functions j' and k at this point. But in the interior without the origin and in the exterior we can go further:

In the interior, if (as in Propositions 2 to 4)

$$\Omega_l \in C^0(]0, R[) \tag{A8}$$

then, in particular, Ω_l is bounded in every compact subinterval, i.e. (A7) is satisfied, and, with the assumption made that ω_l is bounded, (A4), we found before that ω_l is a class $C^{1,1}$ function. Particularly, $\omega_l \in C^0(]0, R[)$, which, together with (A8), yields [cf. (A5)] $I_l \in C^0(]0, R[)$, following, by the second fundamental theorem of calculus, that the integral in the numerator of (A6) is in $C^1(]0, R[)$, on the other hand, also the function j, in the denominator, is in particular class C^1 in $[0, R]$ (interior), and, therefore, ω_l' is in $C^1(]0, R[)$, yielding

$$\omega_l \in C^2(]0, R[)\,.$$

In the exterior (where $j' \equiv 0$) actually, starting from an (almost everywhere) bounded function, assumption (A4), and arguing as before recursively, we even find, by induction, $\omega_l \in C^\infty(]R, \infty[)$, as is known for a solution of an ordinary differential equation with analytic coefficient functions.

So far, for $\Omega_l \in C^0(]0, R[)$ and $\omega_l \in L^\infty([0, \infty[)$, we have found in particular $\omega_l \in C^2(]0, \infty[\backslash \{R\}) \cap C^1(\mathcal{N}_R)$ [$\mathcal{N}_R$, a neighborhood of $r = R$].

Let us then study the regularity at $r = 0$. We make this in general for $\omega \equiv \omega(r, \theta)$, solution of the complete equation ($r \in [0,\infty[,\ \ \theta \in [0,\pi]$) [Eq. (8) divided by r^4],

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 j(r)\frac{\partial\omega}{\partial r}\right] + \frac{k(r)}{r^2 sin^3\theta}\frac{\partial}{\partial\theta}\left[sin^3\theta\frac{\partial\omega}{\partial\theta}\right] + \frac{4}{r}\frac{dj}{dr}(r)\,\omega = \frac{4}{r}\frac{dj}{dr}(r)\,\Omega(r,\theta). \tag{A9}$$

We shall prove

Property (b):

$$\begin{aligned} &\textit{if} \quad \Omega(.,\theta) \in C^{0,\gamma} \ \textit{at}\ r=0,\ \textit{for some}\ \gamma<1 \quad [\omega(.,\theta)\in L^\infty\ \textit{at}\ r=0], \\ &\textit{then} \quad \omega(.,\theta) \in C^{2,\gamma}\ \textit{at}\ r=0\,. \end{aligned} \tag{A10}$$

We first consider the following change of coordinates, by first introducing cylindrical coordinates in the meridian plane,

$$(r,\theta) \longrightarrow (\rho := r\sin\theta\ ,\ \ z := r\cos\theta)\,,$$

and second, making a "5-lift" of ω to $\mathbb{R}^5$. To this end, we identify

$$\mathbf{x} \equiv (x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$$

with

$$\left(\rho = \sqrt{\sum_{i=1}^{4} {x_i}^2}\ ,\ z = x_5\right) \in \mathbb{R}_0^+ \times \mathbb{R}\,,$$

or equivalently with

$$r \equiv r(\mathbf{x}) = \sqrt{\sum_{i=1}^{5} {x_i}^2} \quad \text{and} \quad \theta \equiv \theta(\mathbf{x}) = \arccos\left(\frac{x_5}{\sqrt{\sum_{i=1}^5 {x_i}^2}}\right) \tag{A11}$$

in the original spherical coordinates.

Hence, we define with $\omega(r,\theta)$ the "lifted" function $\tilde\omega$ in flat $\mathbb{R}^5$, axisymmetric around the x_5-axis, by

$$\tilde\omega(\mathbf{x}) \equiv \tilde\omega(x_1,x_2,x_3,x_4,x_5) := \omega\left(r(\mathbf{x}), \arccos\left(\frac{x_5}{r(\mathbf{x})}\right)\right)\,.$$

Now, the flat Laplacian in 5 dimensions of $\tilde\omega$ gives

$$\begin{aligned} \Delta\tilde\omega \equiv \sum_{i=1}^{5}\partial_i^2\tilde\omega &= \partial_r^2\omega + \frac{4}{r}\partial_r\omega + \frac{1}{r^2}\partial_\theta^2\omega + \frac{3\cot\theta}{r^2}\partial_\theta\omega \\ &= \frac{1}{r^4}\partial_r[r^4\partial_r\omega] + \frac{1}{r^2\sin^3\theta}\partial_\theta[\sin^3\theta\,\partial_\theta\omega]\,. \end{aligned}$$

Since, as $r \to 0$, $j(r) = j_0 + j_1 r^2 + \ldots$ and $k(r) = k_0 + k_1 r^2 + \ldots$, with $j_0 = k_0 = j(0) > 0$, and the coefficient of ω and Ω in Eq. (A9) is

$$\frac{4}{r}\frac{dj}{dr}(r) = \frac{4}{r}(2j_1 r + 3j_2 r^2 + \ldots) = 8j_1 + 12 j_2 r + \ldots, \quad \text{where} \quad j_1 < 0\,,$$

we can thus read Eq. (A9) in a neighborhood of $r = 0$ as a Poisson-equation in a 5-dimensional flat space, and $\tilde{\omega}$ will satisfy

$$\Delta\tilde{\omega}(\mathbf{x}) = \frac{8j_1}{j_0}[\tilde{\Omega}(\mathbf{x}) - \tilde{\omega}(\mathbf{x})] \qquad (j_0 > 0,\ j_1 < 0)\,. \tag{A12}$$

Therefore, by the regularity of Poisson's integral,[9] provided that the functions ω and Ω are in L^∞ (in particular in a neighborhood of $r = 0$), it follows that $\tilde{\omega}$, and therefore, through the change of coordinates (A11), also ω, is at $r = 0$ at least a class $C^{1,\gamma}$ function for some $\gamma < 1$, with respect to all their arguments. In particular, (for each θ fixed) at least $\omega(.,\theta) \in C^{1,\gamma}$ in a neighborhood of $r = 0$ (for some $\gamma < 1$). Notice, from the continuity at $r = 0$, its Fourier-Jacobi coefficients (series expansion in $x \equiv \cos\theta$) must satisfy

$$\omega_l(0) = 0 \quad \forall l \geq 2\,,$$

as has been used in Propositions 2 and 3. $\omega_l'(0) = 0\ \forall l \geq 1$ follows directly from Eq. (A3).) Further, assuming (as in Sec. III) that $\Omega(.,\theta)$ is at least a class $C^{0,\gamma}$ (for some $\gamma < 1$) in a neighborhood of $r = 0$ [from where

$$\Omega_l(0) = 0 \quad \forall l \geq 2\ ]\,,$$

we have in particular $\Omega(.,\theta) - \omega(.,\theta) \in C^{0,\gamma}$ in a neighborhood of $r = 0$, and, again, by the regularity of the Poisson integral applied to (A12), $\tilde{\omega}$, and hence ω, is at $r = 0$ a class $C^{2,\gamma}$ with respect to all their arguments; particularly, $\omega(.,\theta) \in C^{2,\gamma}$ in a neighborhood of $r = 0$ (for some $\gamma < 1$).

APPENDIX B: Jacobi-Fourier expansion of $\Omega - \omega$

The series expansions in the orthogonal system [in $L^2_\rho([-1,1])$] of the derivatives of Legendre polynomials (6), $\{y_l\}_{l=1}^\infty \equiv \{\frac{d\mathcal{P}_l}{dx}\}_{l=1}^\infty$, can be transformed into series expansions in Jacobi polynomials with parameters $\alpha = \beta = 1$, $\{\bar{y}_l\}_{l=0}^\infty \equiv \{\mathcal{P}_l^{(1,1)}\}_{l=0}^\infty$. Both systems are indeed equal up to a factor, namely,[10]

$$\frac{d\mathcal{P}_l}{dx} = \frac{l+1}{2}\mathcal{P}_{l-1}^{(1,1)}\,.$$

Accordingly, for each $r \geq 0$ fixed, we consider the series expansions

$$\begin{aligned}
\Omega(r,x) &\sim \sum_{l=1}^{\infty} \Omega_l(r) \frac{d\mathcal{P}_l}{dx}(x) &=& \sum_{n=0}^{\infty} \bar{\Omega}_n(r) \mathcal{P}_n^{(1,1)}(x), & \text{(B1)} \\
\omega(r,x) &\sim \sum_{l=1}^{\infty} \omega_l(r) \frac{d\mathcal{P}_l}{dx}(x) &=& \sum_{n=0}^{\infty} \bar{\omega}_n(r) \mathcal{P}_n^{(1,1)}(x), & \text{(B2)} \\
\text{with } \bar{\Omega}_n &\equiv \frac{n+2}{2} \Omega_{n+1} \quad \text{and} & & \bar{\omega}_n \equiv \frac{n+2}{2} \omega_{n+1} & \text{(B3)} \\
\bar{\lambda}_n &= \lambda_{n+1} = n(n+3) & & (n = 0, 1, \ldots) \, . &
\end{aligned}$$

The orthogonal system of Jacobi polynomials,[3] $\{\mathcal{P}_l^{(1,1)}\}_{l=0}^{\infty}$ is *closed*, and hence *complete*, on the interval $]-1,1[$ with respect to continuous functions in $L^2_\rho([-1,1])$, $\rho(x) := 1 - x^2$. Therefore, convergence in the mean, that is, in the norm $\|.\|_\rho$ in $L^2_\rho([-1,1])$ of the *Jacobi-Fourier* series (B1) and (B2) follows from the particular requirement $\Omega(r,.)$, $\omega(r,.) \in L^2_\rho([-1,1]) \cap C^0([-1,1])$ [as required in Sec. III]; but then also the function difference $\Omega(r,.) - \omega(r,.) \in L^2_\rho([-1,1]) \cap C^0([-1,1])$, and, as a consequence, its Jacobi-Fourier expansion,

$$\Omega(r,x) - \omega(r,x) \sim \sum_{n=0}^{\infty} [\bar{\Omega}_n(r) - \bar{\omega}_n(r)] \mathcal{P}_n^{(1,1)}(x) \, ,$$

will converge also (at least) in the mean.

Furthermore,requiring (as in Sec. III) $\Omega(r,.) \in C^1([-1,1])$ and $\exists \frac{\partial^2 \Omega}{\partial x^2}$ bounded (with respect to x), guarantees that the series (B1) converges uniformly.[11] On the other hand, each function $\bar{\omega}_n$ (for each $n = 0, 1, \ldots$) is solution of a second order differential equation, Eq. (8), with inhomogeneity term $\bar{\Omega}_n$, and therefore its regularity is two orders higher than the regularity of $\bar{\Omega}_n$, that will assure that its series expansion (B2) converges even absolutely. [By Lemma 3 in Appendix C, if $\bar{\Omega}_n = O((n+1)^{-p})$, then $\bar{\omega}_n(\bar{\Omega}_n) = O((n+1)^{-p-2})$. Therefore, since $\bar{\Omega}_n = O((n+1)^{-3/2})$, we have $\bar{\omega}_n(\bar{\Omega}_n) = O((n+1)^{-7/2})$. With this, in particular, since on the other hand $|\mathcal{P}_n^{(1,1)}(x)| \leq \mathcal{P}_n^{(1,1)}(1) = n+1$, it follows $|\bar{\omega}_n(r)| \left|\mathcal{P}_n^{(1,1)}(x)\right| = O(n+1)^{-5/2})$, yielding a series (B2) which converges absolutely].

Notice, assuming equatorial symmetry, i.e. symmetry about the plane $x = 0$ $(= \pi/2)$, $\omega \equiv \omega(r,x)$ is an even function in x; on the other hand, and by the symmetry relation (parity) $\mathcal{P}_n^{(\alpha,\beta)}(-x) = (-1)^n \mathcal{P}_n^{(\alpha,\beta)}(x)$, it follows that when n is even (odd), $\mathcal{P}_n^{(\alpha,\beta)}$ is an even (odd) function in x [$\mathcal{P}_n^{(\alpha,\beta)}$ contains only even (odd) powers of

$x]$; thus, looking at the series expansion (B2) of the function $\omega(r,.)$ even in x, the odd coefficients should vanish, $\bar{\omega}_n \equiv 0 \;\; \forall n \geq 0 \;\; n$ odd, and, by Proposition 1, $\bar{\Omega}_n \equiv 0 \;\; \forall n \geq 0 \;\; n$ odd, as well, that is, by the expansion (B1), Ω is an even function in x. Hence, we have the *even* series expansions

$$\Omega(r,x) = \sum_{k=0}^{\infty} \bar{\Omega}_{2k}(r)\mathcal{P}_{2k}^{(1,1)}(x) \quad \text{and} \quad \omega(r,x) = \sum_{k=0}^{\infty} \bar{\omega}_{2k}(r)\mathcal{P}_{2k}^{(1,1)}(x)\,, \quad \text{yielding}$$

$$\Omega(r,x) - \omega(r,x) \;=\; \sum_{k=0}^{\infty}[\bar{\Omega}_{2k}(r) - \bar{\omega}_{2k}(r)]\mathcal{P}_{2k}^{(1,1)}(x)\,, \tag{B4}$$

converging at least uniformly. Actually, since $\mathcal{P}_0^{(1,1)} = \frac{d\mathcal{P}_1}{dx} \equiv 1$, we have

$$\Omega(r,x) \;=\; \bar{\Omega}_0(r) + \sum_{k=1}^{\infty} \bar{\Omega}_{2k}(r)\mathcal{P}_{2k}^{(1,1)}(x) \;=\; \bar{\Omega}_0(r) + \bar{\Omega}_2(r)\mathcal{P}_2^{(1,1)}(x) + \bar{\Omega}_4(r)\mathcal{P}_4^{(1,1)}(x) + \ldots$$

$$\omega(r,x) \;=\; \bar{\omega}_0(r) + \sum_{k=1}^{\infty} \bar{\omega}_{2k}(r)\mathcal{P}_{2k}^{(1,1)}(x) \;=\; \bar{\omega}_0(r) + \bar{\omega}_2(r)\mathcal{P}_2^{(1,1)}(x) + \bar{\omega}_4(r)\mathcal{P}_4^{(1,1)}(x) + \ldots$$

and therefore

$$\begin{aligned} \Omega(r,x) - \omega(r,x) \;&=\; \bar{\Omega}_0(r) - \bar{\omega}_0(r) + \sum_{k=1}^{\infty}[\bar{\Omega}_{2k}(r) - \bar{\omega}_{2k}(r)]\mathcal{P}_{2k}^{(1,1)}(x) \\ &=\; \bar{\Omega}_0(r) - \bar{\omega}_0(r) + [\bar{\Omega}_2(r) - \bar{\omega}_2(r)]\mathcal{P}_2^{(1,1)}(x) + \ldots \end{aligned} \tag{B5}$$

Notice, condition $\frac{\partial\Omega}{\partial x} \not\equiv 0$ guarantees $\bar{\Omega}_{n_0} \not\equiv 0$ for some $n_0 \geq 2$.

APPENDIX C: Results on Jacobi-Fourier series

Lemma 1 [positivity of Jacobi-Fourier sums]

$$\sum_{n=0}^{N} \frac{\mathcal{P}_n^{(\alpha,\alpha)}(x)}{\mathcal{P}_n^{(\alpha,\alpha)}(1)} \geq 0 \quad \forall \alpha \geq 0 \quad \forall x \in]-1,1] \quad \forall N \geq 0 \quad (N \in \mathbb{N})\,.$$

[cf. Eq. (4.10.18) in Ref. 12, p. 97, or Ref. 13]

In particular, for $\alpha = 1$,

$$\sum_{n=0}^{N} \frac{\mathcal{P}_n^{(1,1)}(x)}{\mathcal{P}_n^{(1,1)}(1)} \geq 0 \;\; \forall x \in]-1,1] \quad \forall N \geq 0,$$

whose even part is also non-negative, that is, due to the parity relation $\mathcal{P}_n^{(1,1)}(-x) = (-1)^n\mathcal{P}_n^{(1,1)}(x)$,

$$\sum_{k=0}^{N} \frac{\mathcal{P}_{2k}^{(1,1)}(x)}{\mathcal{P}_{2k}^{(1,1)}(1)} \geq 0 \;\; \text{holds} \;\; \forall x \in [-1,1] \quad \forall N \geq 0\,,$$

or, since $\mathcal{P}_n^{(1,1)}(1) = n+1$,

$$\sum_{k=0}^{N} \frac{\mathcal{P}_{2k}^{(1,1)}(x)}{2k+1} \geq 0 \quad \forall x \in [-1,1] \quad \forall N \geq 0\,. \tag{C1}$$

Lemma 2 [Abel transformation (summation by parts)]

If

$$\sum_{k=0}^{N} a_k \geq 0 \quad \forall N \;\geq 0 \quad (N \in \mathbb{N}) \tag{C2}$$

$$\textit{and} \quad 0 < b_N \leq b_{N-1} \leq \ldots \leq b_1 \leq b_0\,, \tag{C3}$$

then

$$\sum_{k=0}^{N} a_k b_k \geq 0\,. \tag{C4}$$

Lemma 3

Consider the Jacobi-Fourier expansion of a function, $f(x) = \sum_{n=0}^{\infty} c_n \mathcal{P}_n^{(\alpha,\beta)}(x)$, *if* $f \in C^q([-1,1])$, *then* $c_n = O((n+1)^{-1/2-q})$.

In words, there is a correspondence between smoothness of a function and order of magnitude of its Jacobi-Fourier coefficients; they fall off faster for smoother functions. Specifically, it follows from the last part of Exercise 91 on p. 391 of Ref. 12 that the Jacobi coefficients of a bounded function are $O((n+1)^{-1/2})$. This estimate also holds for L^2_ρ functions via the Parseval relation.[14] Via the differentiation formula (4.21.7) in Ref. 12, it follows that if the function is q-times continuously differentiable, then the estimate $O((n+1)^{-1/2-q})$ holds.[15] In our case, since $\Omega(r,.) \in C^1([-1,1])$ [requirement (ii) in Sec. III], its Jacobi coefficients $\bar{\Omega}_n(r) \equiv \bar{\Omega}_n = O((n+1)^{-1/2-1}) = O((n+1)^{-3/2})$.

Lemma 4

The Fourier-Jacobi coefficients, c_n, *of an absolutely monotonic function [i.e. of a function* f *satisfying* $\frac{d^n f}{dx^n}(x) \geq 0 \;\; \forall n \geq 0$*] are non-negative numbers.*[16] Moreover, in the particular case where f is an even function (of x) it is enough to require $\frac{d^{2k} f}{dx^{2k}}(x) \geq 0 \;\; \forall k \geq 0$ (all even order derivatives ≥ 0).[17]

Notice, condition (i) in Sec. III guarantees (α) of Proposition 3 [via a theorem on parameter-integrals applied to Eq. (7)]; also, by continuity at $r=0$ [cf. (i), Sec. III], $\Omega_l(0) = 0 \;\; \forall l \geq 2$ and, by (30), $\omega_l(0) = 0 \;\; \forall l \geq 2$ as well.

It has been seen, by equatorial symmetry, $\Omega(r,.)$ (and its derivatives with respect to r) are all even functions of x. Therefore, by Lemma 4, requirement (VI) in Sec. III guarantees (β), (γ) and (δ) of Proposition 3 (Sec. II). [Note, $\lambda_l \geq \lambda_3 = 10\ \forall l \geq 3$ ($\bar{\lambda}_n \geq \bar{\lambda}_2 = 10\ \forall n \geq 2$), and, since $k(r)/j(r) > 0$ and $\Omega_l \geq 0$, we have

$$\frac{\lambda_l}{r^2}\frac{k(r)}{j(r)}\Omega_l \geq \frac{10}{r^2}\frac{k(r)}{j(r)}\Omega_l \quad \forall l \geq 3\,].$$

On the other hand, (III) yields $\bar{\Omega}_0'(r) \equiv 0$, i.e. $\bar{\Omega}_0(r) \equiv$ const. and with (IV), $\bar{\Omega}_0(r) =$ const. > 0 $[\Omega_1(r) =$ const. $> 0]$, so that Proposition 5 also applies [notice, since $\bar{\Omega}_n(0) = 0\ \ \forall n \geq 2$, we have $\Omega(0,x) = \bar{\Omega}_0(0)$]. Hence, Propositions 3 and 5 apply, and it follows [with notation (B3)]

$$0 < \bar{\omega}_n(r) < \bar{\Omega}_n(r) \ \ \forall r \in]0,R], \qquad \bar{\omega}_n(0) = \bar{\Omega}_n(0) = 0, \ \ \forall \ \text{ even } \ n \geq 2\,,$$

$0 < \bar{\omega}_0(r) < \bar{\Omega}_0(r)\ \ \forall r \in [0,R]$; which can also be written

$$\begin{aligned} & 0 < \bar{\omega}_{2k}(r) < \bar{\Omega}_{2k}(r) \qquad \forall r \in]0,R] \ \ \forall k \geq 0, \\ & 0 < \bar{\omega}_0(0) < \bar{\Omega}_0(0), \\ \text{and} \quad & \bar{\omega}_{2k}(0) = \bar{\Omega}_{2k}(0) = 0 \qquad \forall k \geq 1\,; \end{aligned}$$

in particular, from the first relation,

$$\bar{\Omega}_{2k} - \bar{\omega}_{2k}(\bar{\Omega}_{2k}) > 0 \ \text{ in } \ ]0,R] \ \ \forall k \geq 0\,.$$

References

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[2] Notice, this is an expansion in *vector* spherical harmonics; as expected, for $g_{t\phi}$ transforms under rotation not like a scalar, but like a component of a vector.

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[4] J.B. Hartle, Ap. J. **150**, 1005 (1967).

[5] In formula (44) of Ref. 4, where a similar calculation for the rigidly rotating case is carried on, a multiplying factor, namely, $j(0)$, has been neglected; S^- and S^+ are actually $-l-2$ and $l-1$. Note, as $r \to 0$, $M(r) \sim r^3$, and hence $\exp[-\lambda(r)] \to 1$, yielding $k(0) = j(0)$.

[6] In the exterior $j \equiv 1$, as already remarked, and it can be easily seen $k(r) \to 1$ as $r \to \infty$; i.e. $k(\infty) = j(\infty)$, this is why these algebraic calculations lead to the same behavior for the fundamental solutions at $r \to \infty$ and at the axis, $r \to 0$. (See Ref. 5).

[7] M.H. Protter and H.F. Weinberger, *Maximum principles in Differential Equations* (Prentice-Hall, Englewood Cliffs, NJ, 1967).

[8] See Sec. III and Appendix A, Property (b).

[9] Consider $\Delta u = f$, if f is an almost everywhere bounded function, $f \in L^\infty(D)$, with compact support, then $u \in C^{1,\gamma}(D)$ for some $\gamma < 1$. See, e.g., O.D. Kellogg, *Foundations of Potential Theory* (Dover, New York, 1953), pp. 151, 139.

[10] Actually, Legendre polynomials are Jacobi polynomials with parameters $\alpha = \beta = 0$, $\mathcal{P}_l = \mathcal{P}_l^{(0,0)}$, and the following general derivative relation for Jacobi polynomials holds, $\frac{d\mathcal{P}_l^{(\alpha,\beta)}}{dx} = \frac{1}{2}(l + \alpha + \beta + 1)\mathcal{P}_{l-1}^{(\alpha+1,\beta+1)}$ [see, e.g., Ref. 3, p. 25, Eq. (6), or Ref. 18, p. 63, Eq. (4.21.7)].

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[15] Remember, $F(n) = O(G(n))$ means, by definition, $|F(n)| \leq \text{const.}\ G(n)\ \forall n \geq n_0$, for some $n_0 \in \mathbb{N}$ and some constant.

[16] A. Lupaş, Rev. Anal. Numer. Teoria Aproximaţiei **3**, 79 (1974).

[17] Because $c_{2k} = \text{const}_k \frac{1}{(2k)!}\frac{d^{2k}f}{dx^{2k}}(x_k)$ where $-1 \leq x_k \leq 1$ and $\text{const}_k = \frac{\langle x^{2k}, \mathcal{P}_{2k}^{(\alpha,\beta)}\rangle_\rho}{\langle \mathcal{P}_{2k}^{(\alpha,\beta)}, \mathcal{P}_{2k}^{(\alpha,\beta)}\rangle_\rho}$. A. Lupaş, (Private Communication).[16]

Capítulo 3

Propiedades de dos soluciones interiores exactas

Dado que nuestra intuición está basada en la teoría newtoniana y puede ser engañosa cuando se aplica a fenómenos relativistas, para conseguir una buena comprensión conceptual de ciertos efectos en relatividad general —como son los efectos de la rotación en fluidos autogravitantes completamente relativistas— puede ser de gran ayuda el estudio e interpretación de soluciones particulares físicamente relevantes de las ecuaciones de campo de Einstein, y la comparación de resultados con los conocidos en el dominio newtoniano

Propiedades geométricas

Resultados newtonianos de los cuales todavía no existe un equivalente relativista son la existencia de un plano ecuatorial (simetría de reflexión) para un fluido autogravitante y en rotación estacionaria, y la simetría esférica en el caso estático, además de resultados clásicos sobre la geometría de la superficie frontera, como son una relación entre velocidad de rotación y oblaticidad versus prolaticidad, o la "convexidad vertical": en el caso newtoniano es bien sabido [14] que las figuras en equilibrio de un fluido en rotación permanente deben ser convexas verticalmente, esto es, cualquier línea recta paralela al eje de rotación corta a la superficie del fluido como máximo en dos puntos; en relatividad general no es conocido ningún resultado equivalente. Además, en el marco newtoniano, al tener en el espacio una métrica plana (euclídea), las propiedades de convexidad del cuerpo contenido en una determinada superficie pueden obtenerse a partir de las propiedades de la curvatura de la superficie [14−16]; sin embargo, en el caso de tener una métrica riemanniana (en relatividad general) esto no es directamente así. Para demostrarlo de forma explícita y para una métrica determinada de manera dinámica por las ecuaciones de Einstein, un ejemplo interesante es la solución de Kramer [8] representando de forma exacta el campo gravitatorio interior debido a un cuerpo de fluido perfecto autogravitante y axisimétrico, en rotación rígida y estacionaria.

OBJETIVOS

Para las soluciones exactas interiores de Wahlquist [7] y de Kramer pretendemos derivar varias propiedades geométricas relacionadas con la forma y la convexidad del cuerpo de fluido y su variación con la rotación, mediante un estudio analítico y numérico de la superficie frontera de presión nula. En especial, para la solución de Kramer —en la que la superficie frontera del fluido tiene una expresión simple en términos de funciones elementales— incluyendo el cálculo de la curvatura gaussiana y un análisis detallado de las geodésicas de la métrica 3-dimensional espacial (tiempo constante).

RESULTADOS Y DISCUSIÓN

Considerando la 2-superficie borde del fluido (superficie de presión nula a un tiempo constante) de la solución de Kramer —superficie que, de los cálculos de la integral de Gauss-Bonet, se demuestra topológicamente esférica— vemos primeramente que, para valores altos de rotación de la fuente, esta 2-superficie borde del fluido desarrolla una curvatura gaussiana *negativa* cerca del ecuador, y que la longitud de los círculos paralelos sobre la superficie cerca del ecuador decrecen con el aumento de la rotación del fluido. Desde un punto de vista euclídeo, esta situación se interpretaría como no-convexidad vertical; sin embargo, haciendo un análisis más detallado del cuerpo 3-dimensional acotado por la 2-superficie, obtenemos el análogo de propiedades newtonianas; en concreto, el comportamiento de las geodésicas espaciales dentro del fluido —y, en particular, la introducción de una familia de geodésicas que generaliza el concepto de líneas rectas paralelas al eje de rotación en el caso newtoniano— indica que sí se dan las propiedades newtonianas en lo referente a "convexidad", demostrando el análogo al resultado newtoniano de Lichtenstein en esta solución particular, **Publicación III** [17].

Además, si medimos distancias físicas (geodésicas) desde el centro del fluido a la superficie frontera, podemos ver que la configuración de Wahlquist es "prolata" para la mayoría de los valores admisibles de los parámetros, y en estos casos la prolaticidad aumenta con velocidades de rotación crecientes, **Publicación IV** [18]; mientras que la configuración de Kramer es "oblata" y tal oblaticidad aumenta incrementando la rotación, **Publicación V** [19]; lo cual indica cierta correlación entre dinámica y geometría.

Propiedades cinemáticas y dinámicas

Una forma de estudiar las propiedades de un espacio-tiempo (o métrica) dado es estudiar las propiedades de movimiento de partículas prueba en el seno del mismo; en el caso en el que la fuente del espacio-tiempo (estacionario y axisimétrico) es un fluido, unas partículas privilegiadas son las del fluido, por lo que un estudio de las propiedades cinemáticas y dinámicas de las mismas resulta de particular interés.

Vemos en la literatura que se han utilizado diferentes métodos con el fin de analizar la cinemática y la dinámica de partículas en rotación alrededor de una fuente de gravedad muy compacta. En estos estudios siempre surgen algunos efectos 'anti-intuitivos' desde un punto de vista newtoniano. Aunque estos efectos ya se ven en su forma más simple en el caso estático (sin rotación) —por ejemplo, en el espacio-tiempo de Schwarzschild un aumento de la velocidad angular de la partícula causa más atracción que repulsión por debajo del radio de la órbita fotónica espacialmente circular—, cuando se estudian espacio-tiempos en rotación estacionaria y axisimétricos, se añade el efecto relativista "dragging" o arrastre, producido por la rotación de la fuente.

Se han desarrollado dos principales formalismos dirigidos hacia la interpretación del movimiento relativista en términos de fuerzas newtonianas [20, 21], con sendas particiones de la 4-aceleración de la partícula (4-fuerza sobre la partícula, con signo opuesto) en componentes gravitatoria, centrífuga, Coriolis, etc. En el primero, la "fuerza gravitatoria" es independiente de la velocidad de la partícula, de forma que el efecto 'anómalo', ya presente en el caso estático, se expresa como un cambio de signo en la "fuerza centrífuga" cerca de la fuente. En este caso, estático, exactamente en las órbitas fotónicas la fuerza centrífuga pasa de ser dirigida hacia afuera ('normal') a ser dirigida hacia adentro ('anómalo'). Cuando el formalismo es extendido a espacio-tiempos estacionarios no-estáticos y axisimétricos [20], una "fuerza de Coriolis" está también presente, haciendo que el cambio de la fuerza centrífuga ocurra no en las órbitas fotónicas. En el otro formalismo [21] la "fuerza gravitatoria" contiene el factor relativista de Lorentz, de forma que el efecto 'anómalo' es adscrito al hecho de que la fuerza gravitatoria (atractiva) supera a todas las otras fuerzas en juego. En estos términos se han analizado soluciones exactas exteriores de las ecuaciones de campo estacionarias no-estáticas y axisimétricas (p. ej., Kerr), o aproximaciones de rotación lenta, pero nunca antes soluciones exactas interiores con estas simetrías.

OBJETIVOS

Se pretende estudiar las propiedades cinemáticas y dinámicas de las soluciones de fluido exactas de Wahlquist y de Kramer —especialmente de la solución de Wahlquist, que realmente es una clase de soluciones, incluyendo la de Kramer como un límite, y que además tiene muchas ventajas, como la existencia de soluciones con presión positiva y decreciente (desde el centro hacia el borde) o la existencia de un límite estático (solución de Whittaker). Este estudio cinemático y dinámico se quiere hacer desde varios puntos de vista.

Primeramente, ver de qué forma depende la aceleración de una partícula prueba en movimiento circular ecuatorial (en el plano del ecuador de la solución) de su velocidad angular, a valores extremos de ésta, y su relación con la órbita fotónica que aparece en la solución (en ambas soluciones) a velocidades altas de rotación de la fuente.

En segundo lugar, aplicar los dos principales formalismos de "fuerzas inerciales" en relatividad general, anteriormente mencionados, a partículas en el plano ecuatorial de la solución considerada; así podremos someter a prueba estas definiciones de fuerzas inerciales en situaciones extremas interiores de rotación rápida y gravedad fuerte, donde los efectos relativistas deberían ser más importantes.

Finalmente, presentar un método para separar "fuerzas rotacionales" y "gravitatorias" sobre partículas del fluido en el ecuador, siguiendo una analogía con el análisis clásico newtoniano: basándonos en la ecuación de Euler, introducir un "peso" de las columnas polar y ecuatorial en relatividad general, y obtener dos posibilidades para la definición de fuerza rotacional, analizando ambas en relación con la velocidad de rotación y la forma de la configuración de fluido, **Publicación V**.

RESULTADOS Y DISCUSIÓN

En ambas configuraciones (Wahlquist y Kramer) se obtiene que, a velocidades de rotación del fluido suficientemente altas, para las que ya aparece una órbita fotónica en el plano ecuatorial del fluido, si consideramos partículas prueba que circulan en el plano ecuatorial, la dependencia de la aceleración de la partícula de su velocidad angular, a valores extremos de la misma, está determinada por la posición de la partícula relativa a la órbita fotónica (como teóricamente se argumentaba en [22]). En concreto, el comportamiento es 'intuitivo' para radios menores que el de dicha órbita, mientras que para valores mayores, las partículas co-rotantes se comportan 'intuitivamente' y las partículas contra-rotantes, 'anti-intuitivamente', **Publicación IV**.

Los citados formalismos de "fuerzas inerciales" —descomposición de aceleración (- fuerza)—, aplicados a partículas del fluido en órbita circular en el plano del ecuador de las soluciones, muestran (cuando la rotación de la configuración es suficientemente rápida) un cambio de signo en la "fuerza centrífuga" para partículas circulando en un radio mayor a un cierto radio crítico, de forma que aumentando la velocidad de rotación del fluido tal radio crítico disminuye, esto es, la órbita crítica se acerca al centro; así, a mayor rotación corresponde una región mayor del fluido con esta 'rara' propiedad de, p. ej., "fuerza centrífuga" dirigida hacia adentro, **Publicación IV**.

Y comparando distintos formalismos concluimos que la forma en la que el efecto 'anómalo' o no-newtoniano se deja ver en términos de fuerzas newtonianas depende de la partición *ad hoc* hecha. De hecho, con la nueva partición (en analogía con la descripción newtoniana) propuesta en **Publicación V**, p. ej., para una partícula en el ecuador de la solución de Kramer, la parte de la fuerza (por definición) dirigida hacia adentro —que es siempre menor que la parte dirigida hacia afuera— crece más rápidamente conforme aumenta la rotación, de forma que la fuerza total (hacia afuera) se hace menor a mayor rotación del fluido (de Kramer); por otra parte, a mayor rotación el fluido se hace más *oblato*.

Esto sugiere una cierta correlación entre dinámica y geometría de la configuración (como se espera desde un punto de vista newtoniano); pero, desde luego, no podemos decir esto directamente, entre otras cosas, porque en el formalismo descrito (en **Publicación V**) estamos suponiendo que el campo gravitatorio en el "polo" no es afectado por la rotación, lo cual no es cierto en relatividad general.

En cualquier caso, interesa medir una magnitud de la aceleración o fuerza total sobre la partícula en el ecuador, que, por cierto, es una propiedad *de frontera*, mientras que la elipticidad de la configuración de fluido, medida con distancias geodésicas desde el centro, es una propiedad *interior*.

Analizando la fuerza o aceleración total de las partículas de fluido en el ecuador de la solución de Wahlquist (configuración con presión positiva y decreciente desde el centro, en un régimen de rotación general) —exactamente, el valor absoluto de la fuerza total proyectada sobre el vector unitario de distancia geodésica ecuatorial—, obtenemos que la fuerza total (hacia adentro) se hace mayor cuando la velocidad de rotación del fluido crece; sin embargo, por otra parte el objeto se hace más *prolato* (la distancia geodésica desde el centro disminuye) conforme aumenta la rotación, condición no-newtoniana [23].

Podemos pues concluir que, contrariamente a lo que ocurre en el caso newtoniano, en el que las propiedades de frontera (fuerza total en el ecuador, convexidad, etc.) están relacionadas con las interiores [14−16], en relatividad general no podemos extrapolar directamente propiedades geométricas y dinámicas de la frontera al interior de la configuración de fluido.

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On some geometric features of the Kramer interior solution for a rotating perfect fluid

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Abstract

Geometric features (including convexity properties) of an exact interior gravitational field due to a self-gravitating axisymmetric body of perfect fluid in stationary, rigid rotation are studied. In spite of the seemingly non-Newtonian features of the bounding surface for some rotation rates, we show, by means of a detailed analysis of the three-dimensional spatial geodesics, that the standard Newtonian convexity properties do hold. A central role is played by a family of geodesics that are introduced here, and provide a generalization of the Newtonian straight lines parallel to the axis of rotation.

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1. Introduction

In the (thus far, unfulfilled) quest for a realistic exact solution in general relativity, representing both the exterior and interior gravitational field generated by a self-gravitating axisymmetric mass of perfect fluid in stationary rotation, the detailed analyses of the features of whatever partial results we already have seem relevant. Specifically, comparison with the known results in the Newtonian domain will improve our intuition within the general relativistic regime.

It is remarkable that there exists a number of treatments based on numerical integration of the field equations, or on approximation schemes valid for small rotation rates (applied, in particular, to the calculation of the shape of the bounding surface of the fluid configurations, or to the analysis of the meaning of centrifugal forces), but, surprisingly, very few exact results, based on the growing wealth of interior exact solutions in the literature (both rigidly and differentially rotating). Here, we analyse some geometric features of one such interior solution, in order to check whether the analogues of some Newtonian properties hold. Remarkably, they do, in spite of the fact that the analysis of the bounding surface $p = 0$ in section 3 seems to point naively to the contrary. A more detailed analysis of the three-dimensional geodesics (sections 4 and 5) shows that standard Newtonian features do indeed hold for the solution under consideration, if some of the Newtonian elements are redefined appropriately. In particular, we introduce in section 5 what we believe is the generalization of straight lines parallel to the rotation axis in the Newtonian case: geodesics whose points have constant azimuthal angle and intersect the equatorial plane orthogonally.

2. The Kramer solution

An exact solution of the Einstein field equations, representing the interior gravitational field of a self-gravitating, axially symmetric, rigidly rotating perfect fluid, was introduced by Kramer[1] and further analysed by himself.[2] The metric can be written as

$$\begin{aligned} 2m\, ds^2 &= [\eta - 1 - b\cos\xi e^{-\eta}]dt^2 + [4(\eta-1) - 4b\cos\xi(e^{-\eta} - e^{-1})]dtd\phi \\ &\quad +[4(\eta-1) - 4b\cos\xi(e^{-\eta} + e^{\eta-2} - 2e^{-1})]d\phi^2 + \frac{d\eta^2}{\eta-1} + \frac{e^{\eta}}{b\cos\xi}d\xi^2\ , \end{aligned} \tag{1}$$

where m and b are positive parameters. The coordinate t is a time coordinate, while ϕ is an azimuthal angle. The spacetime possesses the two commuting Killing fields ∂_t and ∂_ϕ. The axis of rotation is characterized by the equation $\eta = 1$, and there exists a discrete symmetry $\xi \to -\xi$. The invariant set under this symmetry (i.e. points with $\xi = 0$) will be referred to as the *equatorial plane* in what follows, and the point with coordinates $\eta = 1$, $\xi = 0$ as the *centre* of the body. The fluid obeys the following barotropic equation of state

$$\varepsilon + 3p = \frac{2m}{\kappa_0} \ , \tag{2}$$

where p is the pressure, ε is the energy density, and κ_0 is a positive constant. The dependence on η and ξ of the pressure and the energy density is the following:

$$p = \frac{m}{2\kappa_0}(1 + \eta - b\cos\xi e^{-\eta}) \tag{3}$$

$$\varepsilon = \frac{m}{2\kappa_0}(1 - 3\eta + 3b\cos\xi e^{-\eta}) \ . \tag{4}$$

It is rather remarkable that the pressure is harmonic in the (η, ξ) coordinates:

$$p_{\eta\eta} + p_{\xi\xi} = 0 \ . \tag{5}$$

Due to the minimum principle for the corresponding Laplacian, the pressure attains its minimum value at the boundary of the domain of definition in the (η, ξ)-plane. This domain is given by the interior of the region bounded by the line $\eta = 1$ and the curve $p(\eta, \xi) = 0$. As a matter of fact, the pressure has its lowest possible (negative) value at the centre. The boundary value $p = 0$ (the greatest value) defines the boundary of the object. In spite of the pressure being negative inside the body, and growing from the centre to the boundary, the dominant energy condition is satisfied:

$$\varepsilon > 0, \quad |p| < \varepsilon \ . \tag{6}$$

It is remarkable that the boundary $p = 0$ has a relatively simple equation,

$$b\cos\xi - (1 + \eta)e^{\eta} = 0 \ . \tag{7}$$

This, and the fact that (as will be shown in section 4) the integration of the relevant spatial geodesics can be reduced to quadratures, is crucial in our analysis of the geometric features of the solution.

The parameter b is related to the modulus of the vorticity vector at the centre by means of

$$(\omega_\mu \omega^\mu)^{\frac{1}{2}} = \sqrt{\frac{m}{2be}}(b + e) \ . \tag{8}$$

When the requirement is made that the metric have the appropriate signature, as well as the requirement that ∂_t be timelike and ∂_ϕ spacelike, the following inequalities result:[1]

$$\begin{gathered} m > 0 \, , \qquad \eta \geq 1 \, , \qquad b\cos\xi > 0 \, , \\ 1 - \eta + b\cos\xi e^{-\eta} > 0 \, , \qquad (\eta + 1)(2e^{\eta-1} - e^{2\eta-2}) \geq 2 \ . \end{gathered} \tag{9}$$

We shall refer to the intersection of the equatorial plane with the boundary $p = 0$ as the *equator* of the body, and to the region with $\xi > 0$ (respectively, $\xi < 0$) as the *northern* (resp., *southern*) *hemisphere*. Similarly, the intersection of the axis of rotation with the boundary $p = 0$ having $\xi > 0$ will be called the *north pole*; the intersection with $\xi < 0$ will be termed the *south pole*. The object is oblate (in the sense that the polar distance to the centre is less than the distance from one point in the equator to the centre), as calculated in ref. 1.

As b is bounded away from zero, there is no static limit for this solution. One is tempted to interpret this feature in the light of the Newtonian result[3] that the pressure cannot have a minimum at the centre if $\Omega^2 < 2\pi\rho$, where Ω is the angular velocity of the Newtonian fluid body, and ρ the mass density. It would be of interest to find the corresponding result in general relativity.

Finally, let us mention that the acceleration and the vorticity vectors are parallel at the pole, and orthogonal at the equator, as required for symmetry reasons.

3. Geometry of the bounding surface $\{t = \text{constant}, p = 0\}$

The spacetime metric (1) induces the following metric on the two-dimensional surface given by $t =$ constant and $p = 0$:

$$\begin{aligned} 2m\, {ds_2}^2 &= g_{\phi\phi} d\phi^2 + g_{\eta\eta} d\eta^2 \qquad (10) \\ &= [4(\eta+1)(2e^{\eta-1} - e^{2\eta-2}) - 8] d\phi^2 + \frac{b^2(\eta+1) - (3\eta+5)e^{2\eta}}{[b^2 - (1+\eta)^2 e^{2\eta}](\eta^2 - 1)} d\eta^2 , \end{aligned}$$

where we have used the equation for the surface $p = 0$ (7) in order to express the two-dimensional metric as a function of the coordinate η. It is remarkable that (for large enough values of the rotation parameter b) the surface possesses a region around the equator with *negative* Gaussian curvature K [cf. figures 1 and 2].

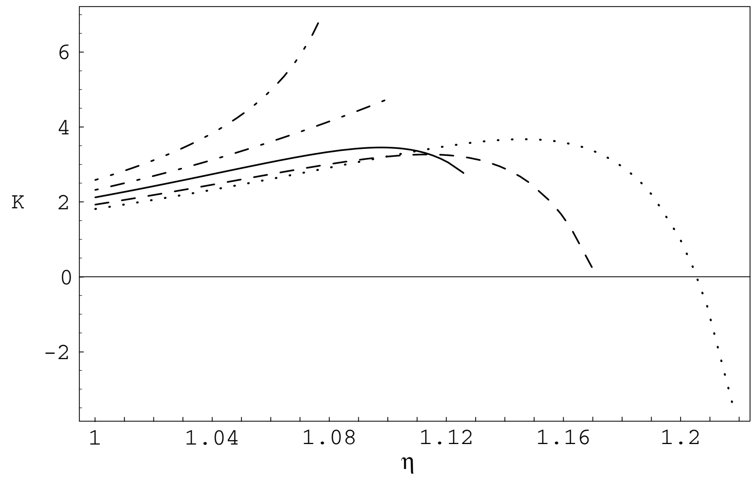

Figure 1. Gaussian curvature on the 2-surface t = constant, $p = 0$, from the pole ($\eta = 1$) to the equator, for different values of the parameter b ($b = 6.1, 6.3, 6.5529, 7, 7.5$) corresponding, respectively, to the double-dot-broken, chain, full, broken and dotted lines in the figure. ($2m = 1$)

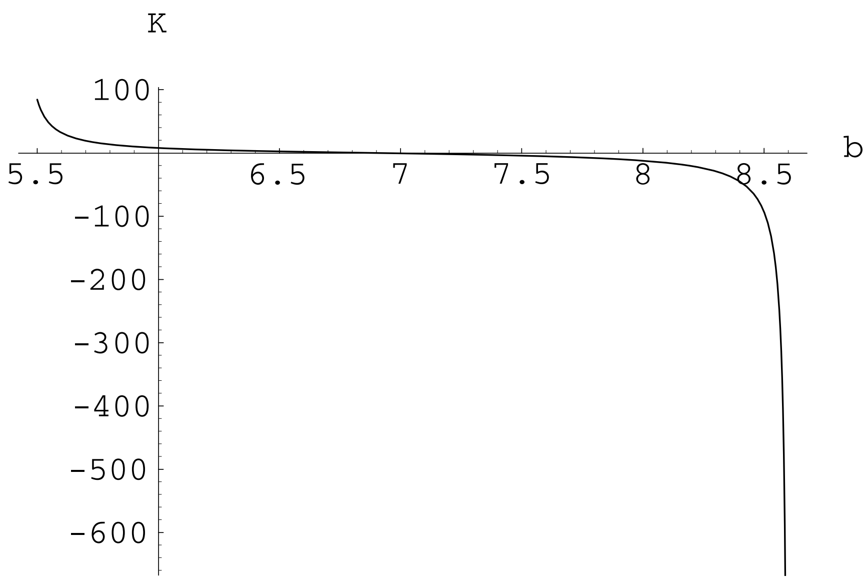

Figure 2. Gaussian curvature on the 2-surface t = constant, $p = 0$, at the equator, as a function of b.

Our intuition with two-dimensional surfaces embedded in three-dimensional Euclidean space would lead us to interpret that region as a concave "waist" near the equator. This impression is reinforced by computing the length of the parallels (closed curves of constant η, that circle around the surface and can be parametrized by means of the azimuthal angle; they are not geodesics, in general). The expression for the length of one such parallel, obtained from (10), is the following:

$$l = \sqrt{\frac{2}{m}}\pi[4(\eta+1)(2e^{\eta-1} - e^{2\eta-2}) - 8]^{\frac{1}{2}} \ . \tag{11}$$

It is easily seen that the length presents a local maximum for

$$2\eta + 4 - (2\eta+3)e^{\eta-1} = 0 \ . \tag{12}$$

As a matter of fact, if the length is plotted as a function of η [cf. figure 3], we see that it increases monotonically from $\eta = 1$ (corresponding to the pole, in which case the length vanishes) to a maximum at $\eta = 1.1716$, obtained by solving (12) numerically. From that point on, the length decreases, until it vanishes again for the numerical value $\eta = 1.3134$. Notice that the preceding values, as well as (12) itself, do not depend on the rotation parameter b. The dependence on b, however, shows up in the following: the coordinate η varies from $\eta = 1$ (intersection of the rotation axis with the surface $p = 0$) to a maximum value (corresponding to the equator), obtained by solving (7) with $\xi = 0$. Accordingly, the maximum value of η is an increasing function of b. When $b < 7.0077$, the corresponding η is such that it falls within the left side of the curve in figure 3, and we have the "normal" situation, where the length of the parallels increases from one pole to the equator. However, if $b > 7.0077$, then the maximum length for a parallel occurs at an intermediate latitude, and it subsequently decreases towards the equator. In the extreme case $b = 8.603$ ($\eta = 1.3134$), the circumferential length at the equator vanishes, which could be interpreted as the fission of the body along the equator at extreme rotation rate. Due to positivity requirements in the metric, the values $\eta > 1.3134$ are excluded.

The Gaussian curvature of the 2-surface vanishes precisely at $\eta = 1.1716$. This is not accidental, as the curvature has a factor $g_{\phi\phi,\eta}$, and $g_{\phi\phi,\eta} = 0$ is precisely the condition expressed by (12). Thus, the surface is "flat" at the equator for $b = 7.0077$. If b is increased, then a finite region with $K < 0$ arises symmetrically around the equator, including the equator itself. At the extreme value $b = 8.603$, K becomes singular (minus infinity, see figure 2) at the equator.

If the 3-geometry where the 2-surface $t =$ constant, $p = 0$ is embedded were Euclidean, we would find that the 3-volume $t =$ constant enclosed by the 2-surface would not be convex, and, in particular, certain straight lines parallel to the axis of rotation would intersect the boundary $p = 0$ in more than two points (for sufficiently

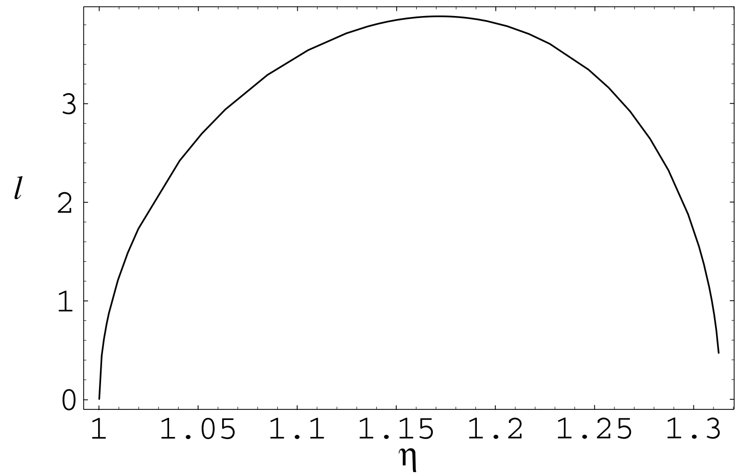

Figure 3. Length of the parallels (closed curves of $\eta =$ constant, parametrized by ϕ in the surface $p = 0$).

large values of b), against the well-known Newtonian theorems of Lichtenstein.[3–5] We shall see below, however, that a natural generalization of the mentioned parallel straight lines to the true three-dimensional Riemannian geometry does preserve the analog of the classical results.

It should be remarked that the closed curves $\eta =$const. on the boundary surface are not geodesics on the surface, except for the particular case where $g_{\phi\phi,\eta} = 0$. This can be readily seen by writing the equations for the geodesics in the metric (10)

$$g_{\phi\phi}\dot{\phi} = \text{constant} \tag{13}$$

$$2g_{\eta\eta}\ddot{\eta} + g_{\eta\eta,\eta}\dot{\eta}^2 - g_{\phi\phi,\eta}\dot{\phi}^2 = 0 \tag{14}$$

(where a dot denotes a derivative with respect to length along the geodesic).

We thus see that the only parallel circles which are geodesics are the two (symmetrically placed with respect to the equator) corresponding to $\eta = 1.1716$, when $b > 7.0077$. When $b = 7.0077$, the two parallels coincide with the equator (and that is the only case in which the equator is a geodesic).

Finally, it is easily shown (by computing the Gauss-Bonnet integral of K numerically over the surface) that the 2-surface $t =$ constant, $p = 0$ has the topology of a 2-sphere.

4. Geodesics of the 3-dimensional spatial metric

The spacetime metric (1) reduces for $t =$ constant to the following three-dimensional metric:

$$2m\, d{s_3}^2 = 4[\eta - 1 - b\cos\xi(e^{-\eta} + e^{\eta-2} - 2e^{-1})]d\phi^2 + \frac{d\eta^2}{\eta - 1} + \frac{e^\eta}{b\cos\xi}d\xi^2 \; . \qquad (15)$$

From (15), we find the following equations for the geodesics in the three-space:

$$[\eta - 1 - b\cos\xi(e^{-\eta} + e^{\eta-2} - 2e^{-1})]\dot\phi = \text{constant} \qquad (16)$$

$$\frac{2\ddot\eta}{\eta - 1} - \frac{\dot\eta^2}{(\eta-1)^2} - \frac{e^\eta}{b\cos\xi}\dot\xi^2 - 4[1 - b\cos\xi(-e^{-\eta} + e^{\eta-2})]\dot\phi^2 = 0 \qquad (17)$$

$$\frac{2e^\eta}{b\cos\xi}\ddot\xi + \frac{2e^\eta\dot\eta\dot\xi}{b\cos\xi} + \frac{e^\eta\sin\xi}{b\cos^2\xi}\dot\xi^2 - 4b\sin\xi(e^{-\eta} + e^{\eta-2} - 2e^{-1})\dot\phi^2 = 0. \qquad (18)$$

In particular, geodesics with $\dot\phi = 0$ are characterized by the two equations

$$\frac{2\ddot\eta}{\eta - 1} - \frac{\dot\eta^2}{(\eta-1)^2} - \frac{e^\eta}{b\cos\xi}\dot\xi^2 = 0 \qquad (19)$$

$$\ddot\xi + \dot\eta\dot\xi + \frac{1}{2}\frac{\sin\xi}{\cos\xi}\dot\xi^2 = 0 \; . \qquad (20)$$

It is a rather remarkable feature of the Kramer solution that the geodesic equations (19) and (20) can be reduced to quadratures. Two clearly different cases appear, depending on whether $\dot\xi = 0$ at all points of the geodesic or not. In the former case, the integration reduces to that of the equation

$$\frac{2\ddot\eta}{\eta - 1} - \frac{\dot\eta^2}{(\eta-1)^2} = 0 \; , \qquad (21)$$

which yields

$$s = q\sqrt{\frac{2}{m}}[\sqrt{\eta_f - 1} - \sqrt{\eta_i - 1}] \; , \qquad (22)$$

where $q = \pm 1$, depending on the sense in which the geodesic is traversed; η_i is the initial η-coordinate, and η_f the final one, and s is the distance along the geodesic. In the case $\dot\xi \neq 0$, we introduce the new variable $w = \sqrt{\eta - 1}$, in order to simplify the equations. By dividing equation (20) by $\dot\xi$, it can be immediately integrated once, giving

$$\frac{\dot\xi^2}{\cos\xi} = ke^{-2\eta} \; , \qquad (23)$$

where k is a positive constant. By substituting (23) into (19), we obtain

$$\ddot{w} - \frac{k}{4be} w e^{-w^2} = 0 \ . \tag{24}$$

If $\dot{w} = 0$, we get the system

$$w = 0 \tag{25}$$

$$\frac{\dot{\xi}^2}{\cos\xi} = k e^{-2} \ . \tag{26}$$

This corresponds to the geodesic along the rotation axis. In the generic case, $\dot{w} \neq 0$, upon multiplication of (24) by $\dot{w}$ we obtain

$$\ddot{w}\dot{w} - \frac{k}{4be} e^{-w^2} w\dot{w} = 0 \ , \tag{27}$$

whose first integral is

$$4\dot{w}^2 + \frac{k}{be} e^{-w^2} = \alpha \tag{28}$$

with $\alpha > 0$ a constant of integration. One can now express the relation among the coordinate w on the geodesic and the distance s along the geodesic by

$$\frac{2dw}{\sqrt{\alpha - \frac{k}{be} e^{-w^2}}} = q\, ds \tag{29}$$

($q = \pm 1$), while the equation for the trajectory is given by

$$\frac{d\xi}{\sqrt{\cos\xi}} = 2\epsilon q \frac{\sqrt{k}}{e} \frac{e^{-w^2}}{\sqrt{\alpha - \frac{k}{be} e^{-w^2}}} dw \ , \tag{30}$$

where $\epsilon = \pm 1$. By using (15), we find $\alpha = 2m$. To summarize, the relevant equations can we written as

$$\dot{w} = q\sqrt{\frac{m}{2}} \frac{1}{\sqrt{\beta}} \sqrt{\beta - e^{-w^2}} \tag{31}$$

$$\dot{\xi} = 2\epsilon q \sqrt{\frac{m}{2}} \frac{1}{\sqrt{\beta}} \sqrt{\frac{b}{e}} \sqrt{\cos\xi}\, e^{-w^2} \tag{32}$$

where $\beta = \frac{be\alpha}{k}$.

Equations (31) and (32) can be expressed as quadratures:

$$\frac{d\xi}{\sqrt{\cos\xi}} = 2\epsilon \sqrt{\frac{b}{e}} \frac{e^{-w^2}}{\sqrt{\beta - e^{-w^2}}} dw \tag{33}$$

$$ds = q\sqrt{\frac{2}{m}} \sqrt{\beta} \frac{dw}{\sqrt{\beta - e^{-w^2}}} \tag{34}$$

(Note that q and ϵ are signs, which can be chosen so that the distance s along the geodesic increases from the initial value $s = 0$ at the initial point.)

5. Convexity properties of the fluid body

Given the results in section 3, one could naively expect that the distance from points in the surface $p = 0$ to the axis of rotation would decrease (for large enough values of b) as the equator is approached. We shall see that this is not the case. In order to do that, let us first identify some general properties of geodesics starting from some point in the symmetry axis and reaching a point in the boundary $p = 0$. For definiteness, we shall work in the northern hemisphere; due to the symmetry with respect to the equatorial plane, analogous considerations hold for the southern hemisphere.

The first observation is that geodesics from the axis of rotation lie in the (w, ξ)-plane, with constant ϕ. This can be seen from equation (16): if the value $w = 0$ $(\eta = 1)$, characterizing the axis, is substituted, then the left-hand side vanishes. Therefore, the constant on the right vanishes. This shows that $\dot{\phi} = 0$ for such geodesics. The relevant equations for the geodesics are then (19) and (20), whose integrals are given by (33) and (34). Next, we find the meaning of the constant β: it is easily seen, by using the standard Riemannian formula, that the angle γ between the axis and the tangent to the geodesic at the axis (defined such that $\gamma = 0$ for a geodesic starting at the axis and pointing towards the north pole) is related to β by the following relation:

$$\cos\gamma = \frac{1}{\sqrt{\beta}} \ . \tag{35}$$

For a given point in the boundary $p = 0$, characterized by $w = w_f$, it is found that a geodesic joining it to the axis has a distance s to the axis given by

$$s = \sqrt{\frac{2}{m}} \int_0^{w_f} \frac{dw}{\sqrt{1 - \cos^2\gamma e^{-w^2}}} \ . \tag{36}$$

From (36), we see that

$$\frac{\partial s}{\partial \gamma} = -\sqrt{\frac{2}{m}} \sin\gamma \cos\gamma \int_0^{w_f} \frac{e^{-w^2}}{(1 - \cos^2\gamma e^{-w^2})^{\frac{3}{2}}} dw \ . \tag{37}$$

Consequently, s is a decreasing function of γ in the northern hemisphere. The minimum is obtained at $\gamma = \frac{\pi}{2}$, which corresponds to a geodesic with constant ξ, whose length (22) is

$$s = \sqrt{\frac{2}{m}} w_f \ . \tag{38}$$

This, being the minimum of the different distances along different geodesics to the axis, we will define as *the distance to the axis* from the given point (w_f, ξ_f) in the boundary. Let us now look at how the distance thus defined varies when we consider different points in the boundary. By using the equation for the boundary,

$$(w_f^2 + 2)e^{w_f^2} = \frac{b}{e}\cos\xi_f \ , \tag{39}$$

and the derivative of w_f with respect to ξ_f along the boundary,

$$\frac{dw_f}{d\xi_f} = -\left(\frac{b}{e}\right)\frac{e^{-w_f^2}}{2w_f^3 + 6w_f}\cos\xi_f \ , \tag{40}$$

we obtain

$$\frac{\partial s}{\partial \xi_f} = \frac{\partial s}{\partial w_f}\frac{dw_f}{d\xi_f} = -\sqrt{\frac{2}{m}}\left(\frac{b}{e}\right)\frac{e^{-w_f^2}}{2w_f^3 + 6w_f}\cos\xi_f \ , \tag{41}$$

thus showing that the distance from a point in the boundary to the axis *increases* monotonically from the north pole to the equator.

Another measure of the convexity (or lack thereof) of the boundary $p = 0$ is the behaviour of the distances from the centre $(w = 0, \xi = 0)$ to points in the boundary [cf. figure 4]. Let us denote by $w_f(\gamma)$ the w-coordinate of the endpoint of a geodesic starting from the centre with an angle γ with respect to the northern semiaxis. The distance to the endpoint will be given by

$$\sqrt{\frac{m}{2}}s = \int_0^{w_f(\gamma)} \frac{1}{\sqrt{1-\cos^2\gamma e^{-w^2}}}dw \ . \tag{42}$$

According to the previous equation, the derivative of s with respect to γ is

$$\sqrt{\frac{m}{2}}\frac{\partial s}{\partial \gamma} = \frac{\partial w_f(\gamma)}{\partial \gamma}\frac{1}{\sqrt{1-\cos^2\gamma e^{-w_f^2(\gamma)}}} - \int_0^{w_f(\gamma)} \frac{\sin\gamma\cos\gamma e^{-w^2}}{(1-\cos^2\gamma e^{-w^2})^{\frac{3}{2}}}dw \ . \tag{43}$$

We are interested in the growth properties of s at the equator. In order to evaluate the derivative $\frac{\partial w_f(\gamma)}{\partial\gamma}$, we consider the equation for the boundary (39), and differentiate it with respect to γ:

$$-\frac{b}{e}\sin\xi_f(\gamma)\frac{\partial \xi_f(\gamma)}{\partial\gamma} = \left[2w_f^3(\gamma) + 6w_f(\gamma)\right]e^{w_f^2(\gamma)}\frac{\partial w_f(\gamma)}{\partial\gamma} \ . \tag{44}$$

The geodesic with $\gamma = \frac{\pi}{2}$ corresponds to a geodesic along the equatorial plane, with $\xi = 0$. Hence, $\xi_f(\frac{\pi}{2}) = 0$. Substituting $\gamma = \frac{\pi}{2}$ in (44), one obtains

$$\left[2w_f^3\left(\frac{\pi}{2}\right) + 6w_f\left(\frac{\pi}{2}\right)\right]e^{w_f^2(\frac{\pi}{2})}\left.\frac{\partial w_f(\gamma)}{\partial\gamma}\right|_{\gamma=\frac{\pi}{2}} = 0 \ . \tag{45}$$

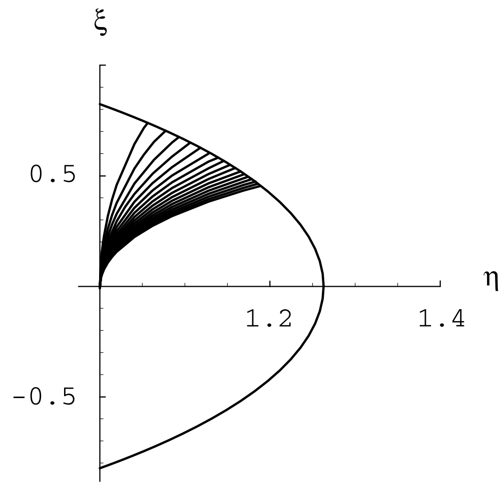

Figure 4. Trajectories of geodesics ϕ = constant from the centre to the surface $p = 0$, with different values of initial velocities, i.e. varying β

However, the coefficient in front of the derivative does not vanish, therefore,

$$\left.\frac{\partial w_f(\gamma)}{\partial\gamma}\right|_{\gamma=\frac{\pi}{2}} = 0 \ . \tag{46}$$

By substituting this result into equation (43), evaluated at $\gamma = \frac{\pi}{2}$, we obtain

$$\left.\frac{\partial s}{\partial\gamma}\right|_{\gamma=\frac{\pi}{2}} = 0 \ , \tag{47}$$

thus showing that the distance from the centre to the equator is a local extremum, if the second derivative of s with respect to γ does not vanish. Its sign will then decide whether the equatorial distance is a maximum (corresponding to local convexity at the equator) or a minimum (local concavity). The second derivative of s with respect to γ will be given from equation (43) by

$$\begin{aligned}
\sqrt{\frac{m}{2}}\frac{\partial^2 s}{\partial\gamma^2} \quad = \quad & \frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}(1-\cos^2\gamma e^{-{w_f}^2(\gamma)})^{-\frac{1}{2}} - (1-\cos^2\gamma e^{-{w_f}^2(\gamma)})^{-\frac{3}{2}} \\
& \times[\cos\gamma\sin\gamma + \cos^2\gamma w_f(\gamma)]e^{-{w_f}^2(\gamma)}\left[\frac{\partial w_f(\gamma)}{\partial\gamma}\right]^2 \\
& -\sin\gamma\cos\gamma e^{-{w_f}^2(\gamma)}(1-\cos^2\gamma e^{-{w_f}^2(\gamma)})^{-\frac{3}{2}}\frac{\partial w_f(\gamma)}{\partial\gamma} \\
& +(\sin^2\gamma - \cos^2\gamma)\int_0^{w_f(\gamma)}\frac{e^{-w^2}}{(1-\cos^2\gamma e^{-w^2})^{\frac{3}{2}}}dw \\
& -3\sin^2\gamma\cos^2\gamma\int_0^{w_f(\gamma)}\frac{e^{-2w^2}}{(1-\cos^2\gamma e^{-w^2})^{\frac{5}{2}}}dw \ .
\end{aligned} \tag{48}$$

At $\gamma = \frac{\pi}{2}$, the preceding equation reduces to

$$\sqrt{\frac{m}{2}}\frac{\partial^2 s}{\partial\gamma^2}\bigg|_{\gamma=\frac{\pi}{2}} = \frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=\frac{\pi}{2}} + \int_0^{w_f(\frac{\pi}{2})} e^{-w^2}dw \ . \tag{49}$$

In order to evaluate the second derivative $\frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}\Big|_{\gamma=\frac{\pi}{2}}$, we first consider the equation for the trajectory of a geodesic from the centre; from (33),

$$\int_0^{\xi_f(\gamma)} \frac{d\xi}{\sqrt{\cos\xi}} = 2\sqrt{\frac{b}{e}}\cos\gamma\int_0^{w_f(\gamma)} \frac{e^{-w^2}}{\sqrt{1-\cos^2\gamma e^{-w^2}}}dw \ . \tag{50}$$

By differentiating (50) with respect to γ, we find

$$\begin{aligned}\frac{\partial\xi_f(\gamma)}{\partial\gamma}\frac{1}{\sqrt{\cos\xi_f(\gamma)}} &= 2\sqrt{\frac{b}{e}}\cos\gamma\frac{\partial w_f(\gamma)}{\partial\gamma}\frac{e^{-w_f^2(\gamma)}}{\sqrt{1-\cos^2\gamma e^{-w_f^2(\gamma)}}} \\ &\quad -2\sqrt{\frac{b}{e}}\sin\gamma\int_0^{w_f(\gamma)}\frac{e^{-w^2}dw}{(1-\cos^2\gamma e^{-w^2})^{\frac{3}{2}}} \ .\end{aligned} \tag{51}$$

Hence

$$\frac{\partial\xi_f(\gamma)}{\partial\gamma}\bigg|_{\gamma=\frac{\pi}{2}} = -2\sqrt{\frac{b}{e}}\int_0^{w_f(\frac{\pi}{2})} e^{-w^2}dw \ . \tag{52}$$

By differentiating (44) we obtain

$$\begin{aligned}&-\frac{b}{e}\cos\xi_f(\gamma)\left[\frac{\partial\xi_f(\gamma)}{\partial\gamma}\right]^2 - \frac{b}{e}\sin\xi_f(\gamma)\frac{\partial^2\xi_f(\gamma)}{\partial\gamma^2} \\ &= \frac{\partial}{\partial w_f(\gamma)}\left\{[2w_f^3(\gamma)+6w_f(\gamma)]e^{w_f^2(\gamma)}\right\}\left[\frac{\partial w_f(\gamma)}{\partial\gamma}\right]^2 \\ &+[2w_f^3(\gamma)+6w_f(\gamma)]e^{w_f^2(\gamma)}\frac{\partial^2 w_f(\gamma)}{\partial\gamma^2} \ ,\end{aligned} \tag{53}$$

and, setting $\gamma = \frac{\pi}{2}$ in (53),

$$-\frac{b}{e}\left[\frac{\partial\xi_f(\gamma)}{\partial\gamma}\bigg|_{\gamma=\frac{\pi}{2}}\right]^2 = \left[2w_f^3\left(\frac{\pi}{2}\right)+6w_f\left(\frac{\pi}{2}\right)\right]e^{w_f^2(\frac{\pi}{2})}\ \frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=\frac{\pi}{2}} \ . \tag{54}$$

Finally, from (54) and (52) we get

$$\frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}\bigg|_{\gamma=\frac{\pi}{2}} = -2\left(\frac{b}{e}\right)^2\frac{e^{-w_f^2(\frac{\pi}{2})}}{w_f^3(\frac{\pi}{2})+3w_f(\frac{\pi}{2})}\left[\int_0^{w_f(\frac{\pi}{2})}e^{-w^2}dw\right]^2 \ . \tag{55}$$

Going back to (49), and substituting $\frac{\partial^2 w_f(\gamma)}{\partial\gamma^2}\Big|_{\gamma=\frac{\pi}{2}}$ from (55), the following expression for the second derivative of the distance is obtained:

$$\sqrt{\frac{m}{2}}\ \frac{\partial^2 s}{\partial\gamma^2}\bigg|_{\gamma=\frac{\pi}{2}} = \int_0^{w_f(\frac{\pi}{2})}e^{-w^2}dw\left[1-\frac{2(\frac{b}{e})^2 e^{-w_f^2(\frac{\pi}{2})}}{w_f^3(\frac{\pi}{2})+3w_f(\frac{\pi}{2})}\int_0^{w_f(\frac{\pi}{2})}e^{-w^2}dw\right] \ . \tag{56}$$

However, substituting $\frac{b}{e}$ from (39) (with $\xi_f = 0$) in (56), we find the following inequality:

$$\frac{2(\frac{b}{e})^2 e^{-w_f{}^2(\frac{\pi}{2})}}{w_f{}^3(\frac{\pi}{2}) + 3w_f(\frac{\pi}{2})} \int_0^{w_f(\frac{\pi}{2})} e^{-w^2} dw = \frac{2[w_f{}^2(\frac{\pi}{2}) + 2]^2 e^{w_f{}^2(\frac{\pi}{2})}}{w_f{}^3(\frac{\pi}{2}) + 3w_f(\frac{\pi}{2})} \int_0^{w_f(\frac{\pi}{2})} e^{-w^2} dw$$
$$\geq \frac{2[w_f{}^2(\frac{\pi}{2}) + 2]^2 e^{-w_f{}^2(\frac{\pi}{2})}}{w_f{}^3(\frac{\pi}{2}) + 3w_f(\frac{\pi}{2})} \int_0^{w_f(\frac{\pi}{2})} e^{-w_f{}^2(\frac{\pi}{2})} dw = \frac{2[w_f{}^2(\frac{\pi}{2}) + 2]^2}{w_f{}^2(\frac{\pi}{2}) + 3} > 2 \ , \tag{57}$$

thus showing that

$$\left.\frac{\partial^2 s}{\partial \gamma^2}\right|_{\gamma=\frac{\pi}{2}} < 0 \ . \tag{58}$$

It should be stressed that (58) does not depend on b. We conclude that the distance from the centre presents a local *maximum* at the equator, thus showing that our naive expectations from the analysis in section 3 were unfounded.

Let us now consider geodesics joining points symmetrically placed with respect to the equator, (w_0, ξ_0) and $(w_0, -\xi_0)$, and having $\dot{\phi} = 0$. For symmetry and differentiability reasons, such geodesics must intersect the equatorial plane orthogonally, with respect to the metric

$$\frac{2}{m} dw^2 + \frac{1}{2m} \left(\frac{e}{b}\right) \frac{e^{w^2}}{\cos\xi} d\xi^2 \ . \tag{59}$$

The orthogonality condition fixes the parameter β in (33) and (34):

$$\beta = e^{-w_c^2} \ , \tag{60}$$

where w_c is the w-coordinate of the intersection of the geodesic with the equatorial plane. We shall now consider the portion of the geodesic starting orthogonally to the equatorial plane from $(w_c, 0)$ and ending in (w_0, ξ_0), whose length will obviously be half that of the complete geodesic starting at $(w_0, -\xi_0)$ and ending in (w_0, ξ_0). The signs in (31) and (34) are fixed by the initial conditions, giving

$$\epsilon q = +1 \ . \tag{61}$$

In principle, the considered portion of geodesic in the northern hemisphere could have $\dot{w} > 0$ or $\dot{w} < 0$. But the latter does not, in fact, exist, as one would have

$$\beta - e^{-w^2} < 0 \ , \tag{62}$$

due to the fact that $w_0 < w_c$ for $\dot{w} < 0$: from (33), such a possibility is incompatible with the equation for the geodesic trajectory. We conclude that the unique geodesic joining $(w_0, -\xi_0)$ and (w_0, ξ_0) cuts orthogonally the equatorial plane at $(w_c, 0)$, with $w_c < w_0$, and exhibits the monotonic behaviour $\dot{w} > 0$ and $\dot{\xi} > 0$ [from equations (61) and (31)] in the northern hemisphere.

Such a geodesic is the natural generalization of a straight line parallel to the rotation axis in the Newtonian case: both can be defined as non-twisting ($\dot{\phi} = 0$) geodesics that intersect the equatorial plane orthogonally. We shall now show that the geodesic thus introduced is completely contained in the three-dimensional body bounded by the surface $t =$ constant, $p = 0$. To this end, we consider the pressure p as a function of a point in the geodesic; from (3),

$$2\frac{\kappa_0}{m}p(s) = w^2(s) + 2 - \frac{b}{e}\cos\xi(s)e^{-w^2(s)} \ . \tag{63}$$

By differentiating (63) with respect to the distances from the starting point $(w_c, 0)$ along the portion of the geodesic in the northern hemisphere (we denote the derivative by a dot), we find

$$2\frac{\kappa_0}{m}\dot{p} = \left(2w + 2\frac{b}{e}\cos\xi we^{-w^2}\right)\dot{w} + \frac{b}{e}\sin\xi e^{-w^2}\dot{\xi} \ ; \tag{64}$$

but, due to the inequalities $\dot{w} > 0$ and $\dot{\xi} > 0$ for the northern portion of the geodesic, and the fact that $2w + 2\frac{b}{e}\cos\xi we^{-w^2} > 0$ and $\frac{b}{e}\sin\xi e^{-w^2} > 0$, we conclude that

$$\dot{p} > 0 \ . \tag{65}$$

As a consequence, the pressure along the geodesic increases from $(w_c, 0)$ to the endpoint (w_0, ξ_0). Conversely, traversing the geodesic in the opposite sense [starting from (w_0, ξ_0) and heading towards $(w_c, 0)$] corresponds to decreasing values of p. As the pressure decreases towards the interior in the solution under consideration, it is clear that a geodesic of the type just introduced which starts at a point in the surface ($p = 0$) in the northern hemisphere has $p < 0$ at all other points in the northern hemisphere. By symmetry, all other points of the geodesic in the southern hemisphere have $p < 0$, except for the final point, where $p = 0$. Thus, a geodesic of the type considered has only two points of intersection with the bounding surface, thus maintaining in the present fluid configuration the classical (Newtonian) result for the intersection of straight lines parallel to the rotation axis with the boundary $p = 0$.

We have found that all the criteria we have considered reproduce the convexity properties of Newtonian configurations, in spite of the peculiar behaviour of the boundary surface, as analysed in section 3. It is clear that the standard Euclidean relations among convex bodies and their bounding surfaces,[6–8] do not hold for a general Riemannian geometry, dynamically prescribed by Einstein's equations.

6. Conclusions

We have analysed some geometric features related to the shape and convexity properties of a self-gravitating body of perfect fluid in stationary rotation, as given by the exact solution in ref. 1. Our analysis of the boundary (t = constant, $p = 0$) shows that for some rotation rates there appear features that would be interpreted in the Euclidean case as non-Newtonian. However, by looking at the behaviour of geodesics in the three-dimensional fluid within the bounding surface, the analog of the Newtonian results under consideration is obtained. The technique to show them is a detailed analysis of the spatial geodesics within the fluid, and specifically the introduction of a family of geodesics which generalize the straight lines parallel to the axis of rotation in the Newtonian case.

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Inertial forces and rotational effects in an interior stationary axisymmetric exact solution

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Abstract

We study several properties (geometrical kinematical and dynamical) of the Wahlquist solution for a rotating object (stationary and axisymmetric). Different approaches to the definition of *inertial forces* in general relativity are analysed for particles in the equatorial plane of this solution (in particular, for fluid particles).

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1. Introduction

The interest of the stationary axisymmetric spacetimes is clear, given that they can describe rotating objects of interest in astrophysics (neutron stars, black holes, etc.). On the other hand, the study of the motion of particles in a given spacetime is a natural method to obtain information from this spacetime. Different approaches have been used in order to study the kinematics and dynamics of particles (particularly, in "circular" motion) in stationary axisymmetric spacetimes.

Our intuition from our "Newtonian" experience drives us to attempt to define generalizations of "inertial forces" for a particle in circular motion in a general relativistic context. Following this idea, we can find several studies defining "inertial forces" for particles in rotation about a very compact source of gravity; in these studies some 'counter-intuitive' effects (from a Newtonian point of view) arise. Although these effects can be seen in their simplest form in the static case [e.g., in the Schwarzschild spacetime an increase of the angular velocity of the particle causes more attraction than repulsion below the radius of the (spatially) circular photon orbit], when studying stationary non-static axisymmetric singularity-free spacetimes the relativistic effect of rotation of the source, i.e. the dragging of the inertial space by mass currents, is added.

Two major formalisms have been developed, aiming at the interpretation of relativistic motion in terms of "Newtonian forces". In the first one,[1,2] based on the "optical reference geometry",[3] the gravitational force is independent of velocity, so that the 'anomalous' effect, already present in the static case, is expressed as a reversal in the direction of action of the "centrifugal force" in the vicinity of the source.[4,5] In this case (static) the reversal occurs exactly at the photon orbits (which are the geodesics of this "geometry"). When the formalism is extended to stationary non-static axially symmetric spacetimes,[6] a "Coriolis force" is also present, making the reversal of the "centrifugal force" not to occur at the photon orbits. Some applications have been made to the Kerr metric,[7] Kerr-Newman,[8,9] and slowly rotating ultracompact objects,[10] where the "eccentricity" behaviour is related to the defined "inertial forces". Gyroscopic precession has also been related to this forces.[6,9,11] In the other approach[12–16] the "gravitational force" contains the relativistic Lorentz factor square, so that the 'anomalous' effect is ascribed to the fact that the (attractive) gravitational force overwhelms all other forces in play. In these terms, the motion of test particles in Kerr-Newman and simpler spacetimes has been analysed,[12,13,17–19] as well as gyroscopic behaviour.[14,20,21]

Our interest in this paper is centered on the study of the properties of interior solutions (perfect fluids) describing rotating compact objects. Very few solutions of this kind (stationary non-static axisymmetric perfect fluid solutions, with a finite boundary of vanishing pressure, satisfying positivity energy conditions and possessing no more Killing vectors than ∂_t and ∂_ϕ) exist; for a revision see, for instance, ref. 22. Among these solutions there is one obtained by Wahlquist, as a special case of a larger family of solutions.[22,23] In order to obtain more information about this solution, we study the kinematical and dynamical properties of the particles in circular motion around the centre of the configuration. Of particular interest are the properties of fluid particles, that will be analysed in more detail (for test particles we can think of them as moving in a closed circular tube around the symmetry axis of the solution, thin enough to not perturb the configuration and allowing the particle to move without interaction with fluid particles).

The rest of the paper is organized as follows. In section 2.1 we briefly revise the Wahlquist solution for rotating fluid bodies, in order to fix our notation. In section 2.2 some geometrical properties of this solution and former studies in the literature on this solution are presented. In section 2.3 we study the total acceleration of a test particle in circular motion; in particular, and following a suggestion given by Semerák,[15] we analyse the dependence of the particle's acceleration on its angular velocity at limiting values, and its relation to the circular photon orbit appearing in the equatorial plane of the Wahlquist solution for certain ranges of the parameter defining the rotation rate. Also, the orbital region is described and the maximally accelerated observer is identified.[16] In section 3, the two formalisms of "inertial forces" in general relativity, mentioned above, are applied to fluid particles in the equatorial plane of the solution under consideration (let us note that, to the best of our knowledge, those formalisms have not been previously applied to interior exact solutions). Hence, we are able to test the definitions of inertial forces in extreme interior situations of rapid rotation and strong gravity, where relativistic effects should be more important. Both formalisms, in the case of circular motion in stationary axisymmetric spacetimes, are briefly introduced, in order to fix the notation, and applied to the Wahlquist solution. In particular, we shall be able to discuss slowly rotating limits and an upper limiting case in rotation, the Kramer solution. The obtained results for fluid particles are analysed in detail and comparison between them is made. This is followed by the concluding remarks of section 4.

2. The Wahlquist solution for rotating fluid bodies. Geometrical and dynamical properties

2.1. The Wahlquist solution

There exists a singularity-free exact interior solution for a finite rigidly rotating body of perfect fluid, as a special case of the axially symmetric, stationary, type-D solution of Einstein's field equations found by Wahlquist[22–24] (in which we set $m = a = 0$). The metric can be written, in terms of comoving pseudoconfocal spatial coordinates (ζ, ξ, ϕ), as:

$$ds^2 = -f(dt - \tilde{A}d\phi)^2 + r_0^2(\zeta^2+\xi^2)\left[\frac{d\zeta^2}{(1-k^2\zeta^2)h_1} + \frac{d\xi^2}{(1+k^2\xi^2)h_2} + \frac{\delta^2 h_1 h_2 d\phi^2}{h_1 - h_2}\right] , \qquad (1)$$

with

$$f \equiv \frac{h_1 - h_2}{\zeta^2+\xi^2} , \qquad \tilde{A} \equiv \delta r_0 \left(\frac{\xi^2 h_1 + \zeta^2 h_2}{h_1 - h_2} - \xi_A{}^2\right) ,$$
$$h_1 \equiv h_1(\zeta) \equiv 1 + \zeta^2 + \frac{\zeta}{\kappa^2}\left[\zeta - \frac{1}{k}(1-k^2\zeta^2)^{1/2}\ \sin^{-1}(k\zeta)\right] ,$$
$$h_2 \equiv h_2(\xi) \equiv 1 - \xi^2 - \frac{\xi}{\kappa^2}\left[\xi - \frac{1}{k}(1+k^2\xi^2)^{1/2}\ \sinh^{-1}(k\xi)\right] ,$$

where κ and k are interior parameters, as well as r_0, δ, and ξ_A, defined below. This solution contains, as special cases, the static limit spherically symmetric (the Whittaker solution) and the Kramer solution, which was shown to be an upper limiting case in rotation.

The fluid has pressure and energy density given by

$$p = \frac{1}{2}\varepsilon_s(1-\kappa^2 f) , \qquad \varepsilon = \frac{1}{2}\varepsilon_s(3\kappa^2 f - 1) ,$$

respectively, and obeys the barotropic equation of state $\varepsilon + 3p = \text{const.} \equiv \varepsilon_s$. We note that the surfaces of constant p, ε and f coincide, and that the constant ε_s is the energy density on the bounding surface $p = 0$, or $\kappa^2 f = 1$.

In general, we have three free parameters, which can be chosen to be the fluid energy density at the bounding surface ε_s, the central pressure p_c, and the modulus of the vorticity at the centre Ω_c —as a “rotation parameter”—, or alternatively, ε_s, k and κ. The parameter r_0 is defined such that $k = \varepsilon_s^{1/2} r_0 \kappa$. The constants ξ_A and δ are adjusted so that the solution behaves properly on the axis: $\xi = \xi_A$ defines the axis of symmetry and rotation, and it is implicitly determined as the solution of the equation $h_2(\xi_A) = 0$, and δ is determined such that the metric satisfies the regularity condition

$$\delta = \pm 2\left[(1+k^2\xi_A{}^2)^{1/2}\ \frac{dh_2}{d\xi}\bigg|_{\xi=\xi_A}\right]^{-1} .$$

We shall set $\varepsilon_s = 1$. The ranges over which the remaining two free parameters (k, κ), or (p_c, Ω_c), run are restricted by positivity and energy conditions, namely, Lorentzian metric, positive pressure, and dominant energy condition.

The surfaces $\zeta = \text{const.}$ and $\xi = \text{const.}$ are the analog of the confocal spheroids and hyperboloids, respectively, in flat space. An important location is the coordinate ring

$(\zeta = 0\ ,\ \xi = 0)$. The ring itself may be either inside $(\kappa^2 < 1)$ or outside $(\kappa^2 > 1)$ the body. Having the ring inside the body implies very slow rotation for normal objects. The metric (1) admits two discret symmetries, $\zeta \to -\zeta$ and $\xi \to -\xi$. If the ring is outside the body, then the invariant set under this symmetry —*equatorial plane*— is the set of points with $\zeta = 0$, whereas if the ring is inside the body, then the equatorial plane will be given by $\{\zeta = 0\ ,\ |\xi| \leq \xi_A\}$ from the axis to the ring and by $\{\xi = 0\ ,\ \zeta > 0\}$ from the ring to the bounding surface of vanishing pressure. The intersection of the equatorial plane with the axis of rotation, i.e. the point with coordinates $(\zeta = 0\ ,\ \xi = \xi_A)$, will be refered to as the *centre* of the body. In the permitted ranges of parameters the general solution has positive and decreasing pressure (from the centre to $p = 0$.) In what follows, the intersection of the equatorial plane with the boundary $(p = 0)$ will be refered to as the *equator* of the body. For definitness, we shall work in the region defined by $\zeta \geq 0$ (the *northern* hemisphere) and $\xi \geq 0$. The intersection of the axis of rotation $(\xi = \xi_A)$ with the boundary $(p = 0)$ having $\zeta > 0$ will be called the *pole* (the *north pole*).

The square of the modulus of the vorticity vector $\Omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\gamma\delta}u_{\nu;\gamma}u_\delta$ (where u^ν is the fluid 4-velocity) at the centre,

$$\Omega_c{}^2 = \frac{1}{4\kappa^2 k^2}\frac{[\kappa^2(\xi_A{}^2 - 1) - k^2\xi_A{}^4]^2}{\xi_A{}^4(1 + k^2\xi_A{}^2)}\ ,$$

will be considered as the rotation parameter.

Up to now, this solution (1) has not been matched to any vacuum solution for a rotating configuration. And there is no definitive proof that it cannot be joined to any asimptotically flat exterior vacuum solution, although some results in this direction have recently been obtained.[25]

2.2. Geometrical properties and definitions

There exist spatial geodesics from the centre along the equatorial plane with $\dot\phi = 0$ and $\dot\zeta = 0$ $(\dot\xi = 0)$, and a geodesic along the rotation axis from the centre to a point in the axis, with $\dot\xi = 0$. Thus, the physical distances from the centre to the surface $p =$ const. are given by the geodesic distances:

$$S_{eq\ (\kappa^2>1)}(\xi) = \int_\xi^{\xi_A} g_{\xi'\xi'}^{1/2} d\xi'$$

in the equatorial plane if the ring is outside,

$$S_{eq\ (\kappa^2<1)}(\zeta) = \int_0^{\xi_A} g_{\xi'\xi'}^{1/2} d\xi' + \int_0^{\zeta} g_{\zeta'\zeta'}^{1/2} d\zeta'$$

if the ring is inside, and

$$S_p(\zeta) = \int_0^\zeta g_{\zeta'\zeta'}^{1/2} d\zeta'$$

along the axis. Their integrands are positive functions, as $g_{\zeta\zeta}$ and $g_{\xi\xi}$ are in the admissible ranges, and the metric is diagonal in ζ and ξ ($g_{\zeta\xi} = 0$); accordingly, we consider their normalized gradients as the equatorial and polar geodesic distance unit vectors:

$$\mathbf{e}_{eq} \equiv \begin{cases} -g_{\xi\xi}^{-1/2}\partial_\xi = \frac{-[(1+k^2\xi^2)h_2]^{1/2}}{r_0\xi}\partial_\xi & \text{if } \kappa^2 > 1 \\ & \text{and from the axis to the ring if } \kappa^2 < 1 \\ g_{\zeta\zeta}^{-1/2}\partial_\zeta = \frac{[(1-k^2\zeta^2)h_1]^{1/2}}{r_0\zeta}\partial_\zeta & \text{from the ring to the surface if } \kappa^2 < 1\,. \end{cases}$$

$$\mathbf{e}_p \equiv g_{\zeta\zeta}^{-1/2}\partial_\zeta\,.$$

Numerically comparing the metric distances from the centre to the bounding surface [i.e. $S_{eq\ (\kappa^2>1)}(\xi_s)$, $S_{eq\ (\kappa^2<1)}(\zeta_s)$, and $S_p(\zeta_p)$, where ξ_s, ζ_s, and ζ_p are solutions of the surface of vanishing pressure $\kappa^2 f = 1$, setting $\zeta = 0$, $\xi = 0$, and $\xi = \xi_A$ respectively], it was shown[26] that the body of fluid described by this solution is *prolate* for most of the admissible values of the parameters, although there is a small range for which it is *oblate*. The referenced study was made fixing the rotation parameter and varying instead the energy density on the bounding surface and a third parameter. However, if we allow the rotation paremeter to be free, it can be seen that, for the range of parameters for which we have prolateness, this solution becomes more and more prolate as the rotation rate increases.

2.3. Dynamical properties

In a stationary axially symmetric spacetime, if we consider a test particle in (spatially) circular motion with 4-velocity

$$\mathbf{u} = u^t\,(\partial_t + \omega\ \partial_\phi)\ , \tag{2}$$

where ω does not depend on t and ϕ (i.e. $u^\mu \omega_{,\mu} = 0$), the 4-acceleration of the particle [or the specific thrust which is required to keep the particle moving (steadily) on a given circular orbit] can be written as

$$a_\mu = \frac{1}{2}\frac{g_{tt,\mu} + 2g_{t\phi,\mu}\ \omega + g_{\phi\phi,\mu}\ \omega^2}{g_{tt} + 2g_{t\phi}\ \omega + g_{\phi\phi}\ \omega^2}\ . \tag{3}$$

In particular, for a fluid particle of the Wahlquist configuration, $\mathbf{u} = u^t \partial_t = g_{tt}^{-1/2} \partial_t$ (i.e. $\omega = 0$ when expressed in comoving coordinates[23]) and

$$a_\mu = \frac{1}{2} \frac{g_{tt,\mu}}{g_{tt}}.$$

Namely,

$$a^\zeta = \frac{1}{2} \frac{d}{d\zeta} \left[\frac{h_1 - h_2}{\zeta^2 + \xi^2} \right] \frac{\zeta^2 + \xi^2}{h_1 - h_2} \tag{4}$$

$$a^\xi = \frac{1}{2} \frac{d}{d\xi} \left[\frac{h_1 - h_2}{\zeta^2 + \xi^2} \right] \frac{\zeta^2 + \xi^2}{h_1 - h_2} , \tag{5}$$

and $a^t = a^\phi = 0$, because of the symmetries.

In the rest of this subsection we shall restrict ourselves to the study of circular motion at the equatorial plane of the Wahlquist solution. We first notice some remarks on the *inward / outward* directions. In the case with the ring coordinate outside the body ($\kappa^2 > 1$), the ζ-component of acceleration in the equatorial plane ($\zeta = 0$) works out to be zero. Obviously, for any $x^\mu(\xi)$ and ω (Ω_c fixed) this acceleration is parallel to the "radial" direction ∂_ξ, $\boldsymbol{a} = a^\xi \partial_\xi$, and for any fixed ξ ($\zeta = 0$) only its magnitude varies with ω. On the other hand, the equatorial geodesic distance (previous subsection) in the case $\kappa^2 > 1$ decreases with the parameter ξ, so that ∂_ξ is an *inward* "radial" direction. Hence, negative a^ξ (as can be numerically seen this is the case for particles of the fluid) means *outward* acceleration (or maintaining thrust) vector.

Also, when the ring is inside the body ($\kappa^2 < 1$) a particle in the equatorial plane between the axis and the ring ($\zeta = 0$) has acceleration $\boldsymbol{a} = a^\xi \partial_\xi$, with $a^\xi < 0$, i.e. *outward* $\boldsymbol{a}$; and a particle in the equatorial plane between the ring and the border ($\xi = 0$) posseses $\boldsymbol{a} = a^\zeta \partial_\zeta$, with $a^\zeta > 0$. Hence, as ∂_ζ is an *outward* "radial" direction, then $\boldsymbol{a}$ is an *outward* vector too.

Therefore, in any case of the Wahlquist configuration, the acceleration of a fluid particle located at the equatorial plane is an *outward* vector.

Notice that in the Wahlquist solution, possesing positive and decreasing pressure from the centre to the bounding surface, the gradient of pressure goes in the *inward* direction (outward "strenght of pressure") (see section 2.1), balancing the *outward* acceleration of equatorial fluid particles, equations (4) and (5), via the Euler equation

$$a_\mu = \frac{-p_{,\mu}}{\varepsilon + p} .$$

In a recent paper Semerák[15] has analysed in a general stationary axisymmetric spacetime the dependence of the 4-acceleration of a circular orbiting test particle on its angular velocity at limiting values (permitted for a time-like orbit) of the velocity. It is argued that the *rotospheres* [defined by the author as the regions where the 4-acceleration (or maintaining thrust) of a given family of orbits diverges to an *outward* direction at the limiting possible values of the orbital angular velocity ω, i.e. where this dependence is counter-intuitive from a Newtonian point of view] are always bounded by photon orbits. We have studied the dependence $a^\xi(\omega)$, given by (3) —equatorial circular motion of test particles— at various fixed "radii" ξ in the Wahlquist field with $\kappa^2 > 1$ (for a fixed rotation rate) at limiting values of ω [cf. figure 1].

Let $\omega_{\rm min}$ and $\omega_{\rm max}$ denote the minimal and the maximal angular velocities which are permitted for a time-like orbit, i.e. the light angular velocities for the counter-rotating (retro-grade) and for the co-rotating (pro-grade) directions respectively,

$$\omega_{\rm min\,,\,max} = \frac{-g_{t\phi} \mp \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} \ .$$

In the Wahlquist spacetime for a rotating fluid body with $\kappa^2 > 1$, for certain intervals of the admissible ranges of parameters, there is a circular photon orbit (light-like geodesic) at the equatorial plane. Let ξ_{ph} denote the "radius" of the photon orbit. It can be numerically seen that at the equatorial plane of the solution under consideration, for any given pair of parameters for which the photon orbit exists, $a^\xi(\omega)$ behaves 'intuitively' *below* the photon orbit ($\xi_{ph} < \xi < \xi_A$) [for both ultrarelativistic co-rotating ($\omega \to \omega_{\rm max}$) and counter-rotating ($\omega \to \omega_{\rm min}$) particles], whereas *above* the photon orbit ($\xi_s < \xi < \xi_{ph}$), it behaves 'intuitively' for $\omega \to \omega_{\rm min}$ and 'counter-intuitively' for $\omega \to \omega_{\rm max}$; that is, for any fixed $\xi > \xi_{ph}$ (below the photon orbit) the co-rotating particle needs greater and greater inward thrust as its velocity approaches that of light; for $\xi < \xi_{ph}$ (above the photon orbit), however, the co-rotating particle needs greater and greater *outward* thrust as its velocity approaches that of light. At the lower limit $\omega \to \omega_{\rm min}$ (counter-rotating particle) the behaviour is 'classical' (intuitive) everywhere. Loosely speaking, one can then conclude that the 'classical' region is below the photon orbit.

At each $\xi > \xi_{ph}$ (below the photon orbit), that is, in the *orbital region*,[16] a^ξ has a positive maximum at some $\omega > 0$ (there exist the *extremely (maximally) accelerated observer*[15,16]) and it is zero for two values of ω (giving the *stationary observers* with $a^\xi = 0$). An example of this behavior is illustrated in figure 1. This dependency of the

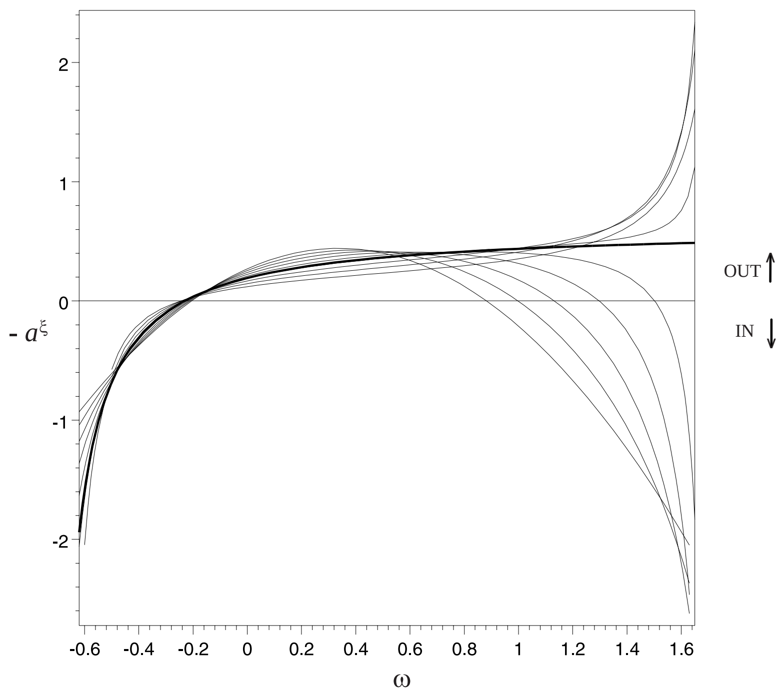

Figure 1. The dependence of (minus) the ξ-component of 4-acceleration, $-a^\xi$, on the angular velocity ω with which a test particle moves on a circular orbit in the equatorial plane of the Wahlquist fluid configuration with $\kappa^2 > 1$, for central pressure $p_c = 0.152$ and rotation parameter ${\Omega_c}^2 = 0.12$, (for which there is one photon orbit at the equatorial plane, corresponding to the thickest line), at various fixed 'radii' ξ. Note: ∂_ξ points in the direction contrary to the increasing geodesic distances, so that $-a^\xi$ has the same sign than the projection of $\boldsymbol{a}$ onto $\mathbf{e}_{eq}$.

4-acceleration of a circularly orbiting test particle on its angular velocity was also analysed for the Kramer solution.[27,28] This configuration, although possesing the undesirable property of negative pressure, satisfies positivity energy conditions and might also provide an insight on fully relativistic rotational aspects and phenomena.

Contrary to what happens in the general Wahlquist solution, in the Kramer solution, the outward gradient of pressure balances the *inward* acceleration of fluid particles. And, similarly to what occurs in the Wahlquist solution, it is below the photon orbit in the equatorial plane —which exits for a certain interval of the permitted range of the rotation parameter b— where the behaviour is also 'intuitive', whereas above the photon orbit it is 'intuitive' for limiting co-rotating test particles but 'counter-intuitive' for limiting counter-rotating particles [cf. figure 2].

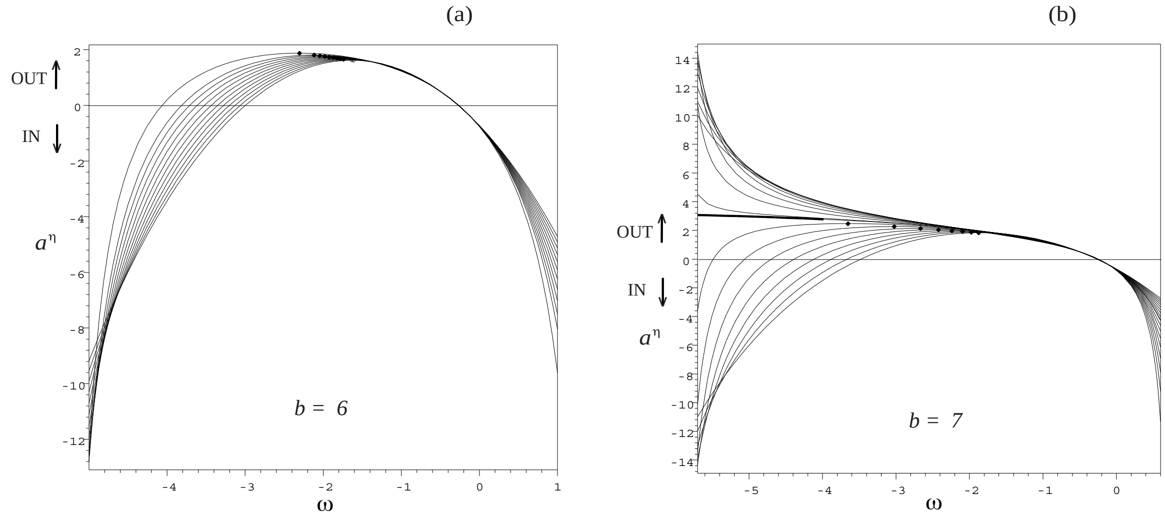

Figure 2. The dependence of the 'radial' component of 4-acceleration a^η on the angular velocity ω with which a test particle moves on a circular orbit in the equatorial plane of the Kramer fluid configuration, at various fixed 'radii' η, with (a) $b = 6$ —when there is no photon orbits—, and (b) $b = 7$ —when there is one photon orbit (corresponding to the thickest line in the figure). Small rhombus indicate the position of the local extrema (maximum) of the curves [the 'extremely (maximally) accelerated observer'[15,16]].

3. "Inertial forces" on fluid particles in the equatorial plane

Aiming at the interpretation of relativistic motion in strong gravitational fields in terms of Newtonian forces, two major formalisms, which maintain the idea of splitting the inertia into different kinds of "inertial forces", have been developed: one, given by Abramowicz and coworkers,[1,6] and based on the *optical reference geometry*,[3] has the gravitational force independent of velocity; the other decomposition, given by Semerák[13] (in accordance with the one proposed by de Felice[12,19]) has the gravitational force containing the relativistic Lorentz factor square. Here we summarize both Abramowicz's[6] and Semerák's[13] decompositions for 'circular' (azimuthal) motion in stationary axially symmetric spacetimes and apply them to equatorial 'circular' motion in the Wahlquist solution for rotating objects.

In stationary axisymmetric spacetimes, the hypersurface-orthogonal (HSO) congruence in the 2-plane (t,ϕ) generated by the symmetries of the space can be shown to be unique, corresponding to the congruence of zero-angular-momentum observers (ZAMOs), let n^μ be their unit 4-velocity;

$$\mathbf{n}^\uparrow = n^t(\partial_t + \omega_d \partial_\phi), \qquad \omega_d \equiv \frac{-g_{t\phi}}{g_{\phi\phi}} \ , \tag{6}$$

that is, $\mathbf{n}_\downarrow = n_t dt$ with the normalization factor

$$n_t = (n^t)^{-1} = -(-g^{tt})^{-1/2} = -\left(\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right)^{-1/2} . \tag{7}$$

The two formalisms that we shall be going through define quantities by projection of their respective decompositions of 4-acceleration onto the 3-space of the ZAMO.

Besides, Abramowicz *et al.*[6] consider, in a general spacetime, the globally hypersurface-orthogonal timelike unit vector n^μ having 4-acceleration equal to the gradient of a scalar function Φ, (i.e. $n^\nu \nabla_\nu n_\mu = -\Phi_\mu$), condition which is fulfilled in stationary axisymmetric spacetimes by the ZAMO [cf. (6),(7)] * with $n_t = -e^{-\Phi}$ $[e^{2\Phi} = -g^{tt}, \ n^t = e^{\Phi}]$. The *optical geometry* is obtained by conformal rescaling of the projected space, $h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu, \ \ \tilde{h}_{\mu\nu} \equiv e^{2\Phi} h_{\mu\nu}$.

Let us now consider a test particle of rest mass m moving in azimuthal 'circular' motion, in our stationary axisymmetric spacetime, with 4-velocity u^μ and 4-acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$. At each point of the trajectory, u^μ can be uniquely decomposed in terms of the ZAMO 4-velocity n^μ,

$$u^\mu = \gamma(n^\mu + v\tau^\mu)$$

where τ^μ is a unit space-like vector orthogonal to n^μ [for azimuthal motion, $\tau^\uparrow = \tilde{r}^{-1}\partial_\phi = (g_{\phi\phi})^{-1/2}\,\partial_\phi$] along which the particle moves with velocity v (with respect to the observer n^μ)

$$v = \tilde{R}\tilde{\omega},$$

where

$$\tilde{R} \equiv n^t g_{\phi\phi}^{1/2} = \frac{g_{\phi\phi}}{(g_{t\phi}^2 - g_{tt}g_{\phi\phi})^{1/2}} \, , \qquad \tilde{r}^2 = g_{\phi\phi}$$

[$\tilde{r} \equiv$ the circumferential radius, which determines the proper circumference of the orbit, $2\pi g_{\phi\phi}^{1/2}$],

$$\tilde{\omega} \equiv \omega - \omega_d \, ,$$

*Notice that if $n_\nu n^\nu = -1$, the minus sign in eq. (7) must hold, instead of what is stated in ref. 6.

and γ is the overall normalization factor, $\gamma^2 = \frac{1}{1-v^2}$. It follows that the 4-velocity of the test particle is $\mathbf{u}^\uparrow = \gamma n^t(\partial_t + \omega\partial_\phi)$, where $\omega \equiv \frac{d\phi}{dt}$ is the coordinate orbital angular velocity of the particle (angular velocity with respect to a static observer ∂_t at ∞). In both formalisms a certain particular "gauge" is adopted, in order to extend τ^μ into a vector field and be able to calculate $n^\nu\nabla_\nu\tau^\mu$.

Once the 4-velocity of a particle, u^μ, is specified, the ZAMO vector field n^μ can be used for the definition of Newtonian analogous "gravitational" and "inertial forces" in stationary axisymmetric spacetimes. In particular, for azimuthal circular motion we have the following splitting formulae:

3.1. Abramowicz's splitting formula

Following Abramowicz's description, the "real force" $f^\mu \equiv ma^\mu$ is balanced by the sum of four forces: "gravitational" G^μ, "Coriolis" C^μ, "centrifugal" Z^μ, and "Euler" E^μ [plus the term $m\dot{\gamma}n^\mu$, which vanishes when projected onto the HSO orbserver (ZAMO) 3-space], i.e. $f^\mu = -G^\mu - C^\mu - Z^\mu - E^\mu + m\dot{\gamma}n^\mu$. The corresponding acceleration decomposition, projected (onto the HSO 3-space) and, once adopted the "Abramowicz-Carter-Lasota-gauge", for circular motion in a stationary axisymmetric spacetime reads $a^\mu = g^\mu + c^\mu + z^\mu + e^\mu$, with

$$g_\mu = a_{\mathrm{ZAMO}\mu} = -\Phi_{,\mu} \tag{8}$$

$$c_\mu = \gamma^2 v\tilde{R}\,\omega_{d,\mu} \tag{9}$$

$$z_\mu = -\gamma^2 v^2\frac{\tilde{R}_{,\mu}}{\tilde{R}} = -\gamma^2 v^2\left(\frac{1}{2}\frac{g_{\phi\phi,\mu}}{g_{\phi\phi}} - a_{\mathrm{ZAMO}\mu}\right) \tag{10}$$

$$e_\mu = 0\,, \tag{11}$$

where $a_{\mathrm{ZAMO}\mu}$ is the ZAMO 4-acceleration. In fact, the term z_μ involves the *geodesic curvature* (in the *optical geometry*)

$$\tilde{\tau}^\nu\tilde{\nabla}_\nu\tilde{\tau}_\mu = -\frac{\tilde{R}_{,\mu}}{\tilde{R}}\,, \qquad \tilde{\tau}^\mu \equiv n_t\tau^\mu = (-g^{tt})^{-1/2}\,\tau^\mu\,.$$

3.2. Semerák's splitting formula

Semerák suggests the decomposition of the 4-acceleration consisting of the "gravitational" part $\mathrm{a}_g{}^\mu$, the "dragging" part $\mathrm{a}_d{}^\mu$, the "Coriolis" part $\mathrm{a}_c{}^\mu$, the "centrifugal" part (minus "normal component of the particle's specific inertial resistance") $\mathrm{a}_z{}^\mu$, and the (minus) "tangent component of the particle's specific inertial resistance" ${a_{ti}}^{\mu}$, which, projected onto the HSO (ZAMO) 3-space, in the stationary and axisymmetric case reads $a^{\mu} = {a_g}^{\mu} + {a_d}^{\mu} + {a_c}^{\mu} + {a_z}^{\mu} + {a_{ti}}^{\mu}$, with

$$a_{g_\mu} = \gamma^2 a_{\mathrm{ZAMO}\mu} \tag{12}$$

$$a_{d\text{-}c_\mu} \equiv 2a_{c_\mu} = 2a_{d_\mu} = 2\gamma^2 v\ (\Omega_{\mathrm{LNRF}} \times \tau^{\uparrow})_\mu = \gamma^2 v \tilde{R}\ \omega_{d,\mu} \tag{13}$$

$$a_{z_\mu} = \gamma^2 v^2 \frac{\Gamma_{\mu.\phi\phi}}{g_{\phi\phi}} = -\frac{1}{2}\gamma^2 v^2 \frac{g_{\phi\phi,\mu}}{g_{\phi\phi}} \tag{14}$$

$$a_{ti_\mu} = 0\,, \tag{15}$$

where $(\Omega_{\mathrm{LNRF}} \times \tau^{\uparrow})_\mu = n_\sigma {\epsilon^\sigma}_{\mu\rho\nu} \Omega^\rho_{\mathrm{LNRF}} \tau^\nu$ if the locally non-rotating frame (orthonormal frame adapted to the ZAMO) rotates with respect to a gyroscope (Fermi-Walker transported) with angular velocity $\Omega^\rho_{\mathrm{LNRF}}$ and ${\epsilon^\sigma}_{\mu\rho\nu}$ is the Levi-Civita tensor. We have denoted $a_{d\text{-}c_\mu} \equiv a_{d_\mu} + a_{c_\mu}$ (in this case $a_{c_\mu} = a_{d_\mu}$).

Analogously to the Abramowicz's formula, the Semerák's centrifugal term can be expressed in the classical Huygens' form, $a_{z_\mu} = \gamma^2 v^2 N_\mu$, where

$$N_\mu = \tau^\nu \nabla_\nu \tau_\mu = -\frac{\tilde{r}_{,\mu}}{\tilde{r}} = \frac{-(g_{\phi\phi}^{1/2})_{,\mu}}{g_{\phi\phi}^{1/2}} = -[\ln(g_{\phi\phi}^{1/2})]_{,\mu} = -\frac{1}{2}\frac{g_{\phi\phi,\mu}}{g_{\phi\phi}} = \frac{\Gamma_{\mu.\phi\phi}}{g_{\phi\phi}}\,.$$

Comparison between the Abramowicz and Semerák's splitting formulae (for 'circular' motion in stationary axisymmetric spacetimes) shows that the Abramowicz's Coriolis part is the Semerák's dragging plus Coriolis parts, $c_\mu = a_{d\text{-}c_\mu}$ ($\equiv a_{d_\mu} + a_{c_\mu}$). As a consequence, the Abramowicz's gravitational plus centrifugal part equals the Semerák's one, $g_\mu + z_\mu = a_{g_\mu} + a_{z_\mu}$.

We analysed both decompositions of the acceleration (projected onto the equatorial geodesic distance unit vector $\mathbf{e}_{eq}$) *for a fluid particle* moving circularly on the equator of the Wahlquist configuration for a rotating fluid, when varying the rotation parameter ${\Omega_c}^2$, for slowly rotating bodies ($\kappa^2 < 1$) [cf. figures 3(a) and 3(c)]; for the general case ($\kappa^2 > 1$) [figures 3(b) and 3(d)]; and for the Kramer configuration [cf. figure 4].

Our first observation is that in the slow rotation limit [figures 3(a) and 3(c)], the behaviour of these 'Newtonian inertial forces' is classical; in particular, we have inward "centrifugal accelerations" (outward "centrifugal forces"). When rotation is important [cf. figures 3(b) and 3(d)], they change sign in the Abramowicz' splitting at ${\Omega_c}^2 \approx 0.085$ [cf. figure 3(a)], where

$$a_{\mathrm{ZAMO}}|_{\xi=\xi_s} - \frac{1}{2}\frac{g_{\phi\phi,\xi}}{g_{\phi\phi}}|_{\xi=\xi_s} = 0\,,$$

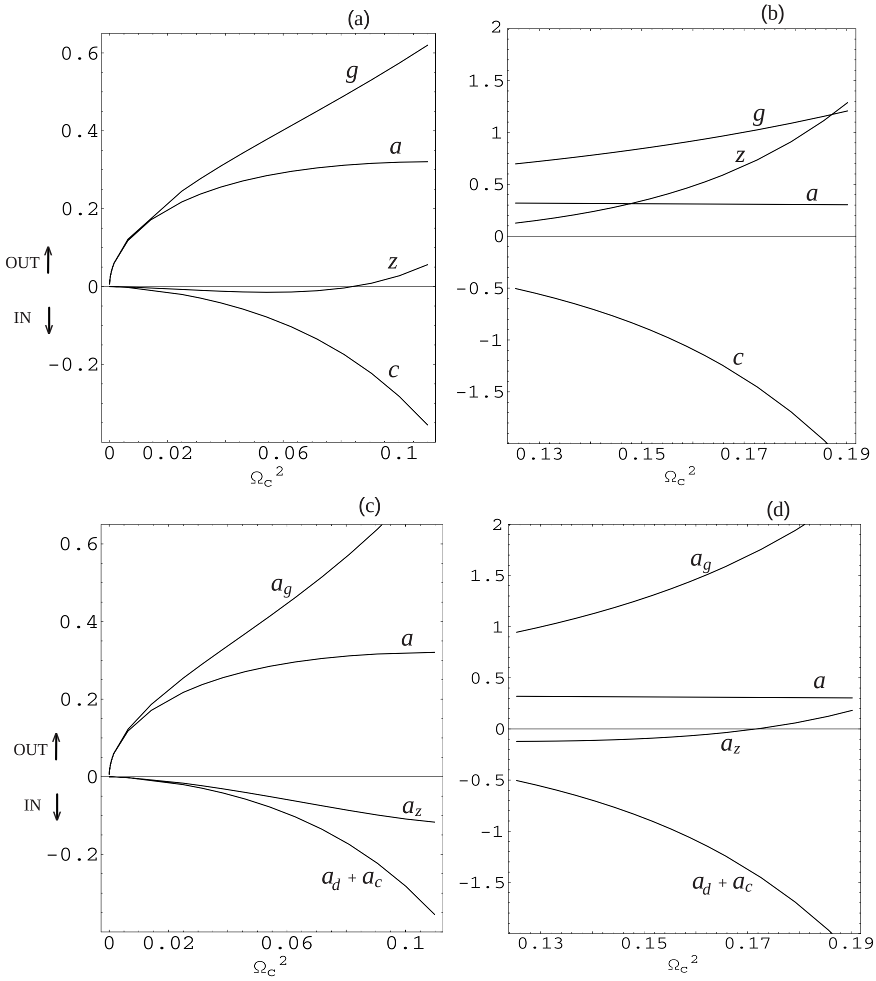

Figure 3. Splitting of the acceleration of a fluid particle in the equator of Wahlquist solution for slow rotation ($\kappa^2 < 1$), in Abramowicz, (a), and Semerák's, (c), formalisms, and for important rotation rates ($\kappa^2 > 1$), in Abramowicz, (b), and Semerák's, (d), cases. (Here 3-vectors acceleration are projected onto the equatorial geodesic distance unit vector $\mathbf{e}_{eq}$).

whereas in the Semerák's one, a similar change of sign occurs at ${\Omega_c}^2 \approx 0.17$ [cf. figure 3(d)], where $a_{\rm ZAMO}|_{\xi=\xi_s} = 0$. One can observe, in both decompositions, that for small rotation rates the "gravitational" part of acceleration dominates over the "centrifugal" plus "Coriolis" ("dragging-Coriolis") part, making positive (outward) the total acceleration, and in this regime the total acceleration increases with the rotation rate; for larger values of the rotation parameter, despite the "centrifugal" part becomes positive (inward "centrifugal force"), the effect of the "Coriolis" part becomes more important, making the total acceleration decrease as the rotation rate increases [figure 3].

In the Kramer case [cf. figure 4] a similar change of sign in the "centrifugal" parts occurs and, in both decompositions, for small rotation rates, the "centrifugal" plus "Coriolis" ("dragging-Coriolis") part of acceleration dominates over the "gravitational" part, making negative (inward) the total acceleration; for larger rotation rates, although the "centrifugal" part becomes positive, the "Coriolis" ("dragging-Coriolis") part dominates again maintaining negative (inward) the total acceleration.

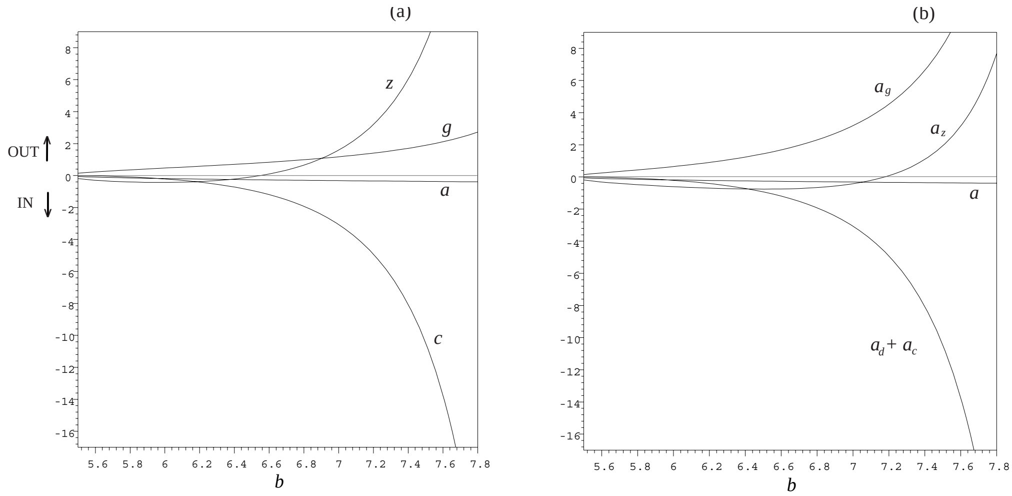

Figure 4. Abramowicz (a) and Semerák's (b) splittings of the acceleration of a fluid particle in the equator of the Kramer solution. 3-vectors acceleration are projected onto the equatorial geodesic distance unit vector $\mathbf{e}_{eq}$ and plotted as functions of the rotation parameter b.

Abramowicz and Semerák's splittings of acceleration for all particles of the fluid orbiting in the equatorial plane of the Wahlquist solution with $\kappa^2 > 1$ was also analysed [cf. figure 5]. Here 3-vectors were projected onto the equatorial geodesic distance unit vector $\mathbf{e}_{eq} = -g_{\xi\xi}^{-1/2}\partial_\xi$, and were plotted as functions of the geodesic (metric) equatorial distance s for several values of the rotation parameter. The result was that in both decompositions, both the (outward) "gravitational acceleration" [inward "gravitational force"] and the (inward) "Coriolis (dragging plus Coriolis) acceleration" [outward "Coriolis force"] increase (in absolute value) from the centre towards the equator. We can observe that the Semerák's "gravitational acceleration" is larger than the Abramowicz's one, and, as already shown, the Semerák's "dragging plus Coriolis accelerations" equals the Abramowicz's "Coriolis acceleration". The behaviour of the "centrifugal acceleration" is also similar in both formalisms; it is negative [positive *outward* "centrifugal force"] for all fluid particles at a certain range of ${\Omega_c}^2$, becomes zero at the equator for ${\Omega_c}^2 \approx 0.085$ in the Abramowicz's definition and for ${\Omega_c}^2 \approx 0.17$ in the Semerák's one, and a region with positive value [negative *inward* "centrifugal force"] arises from the equator to inside the body, above these critical values of ${\Omega_c}^2$.

The same analysis was made for the Kramer solution, giving qualitatively equal results.

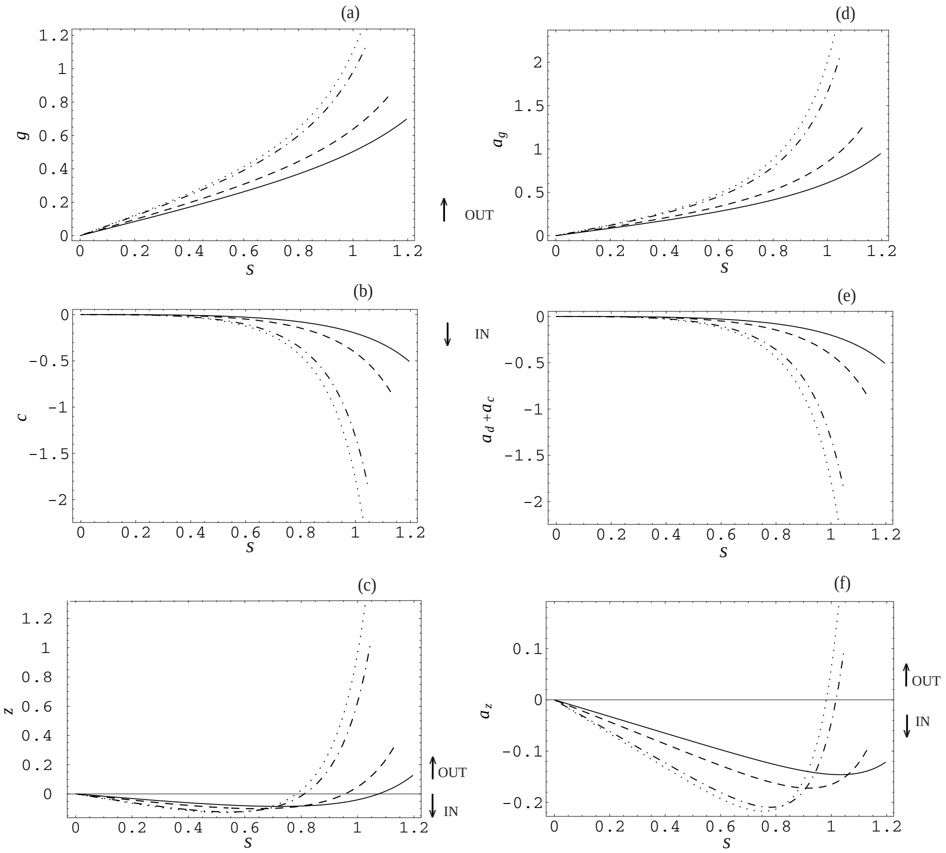

Figure 5. Abramowicz's "gravitational" (a), "Coriolis"(b), and "centrifugal"(c) and Semerák's "gravitational"(d), "dragging plus Coriolis"(e), and "centrifugal"(f) parts of acceleration (projected onto $\mathbf{e}_{eq}$) of fluid particles in the equatorial plane of the Wahlquist configuration (for $\kappa^2 > 1,\ p_c = 0.152$), as functions of the metric equatorial distance s, from the axis of rotation to the equator, for different rotation rates (${\Omega_c}^2 = 0.12, 0.15, 0.18, 0.19$), corresponding, respectively, to the full, broken, chain, and dotted curves in the figure. Notice that the end points of the plotted curves $[\xi = \xi_s({\Omega_c}^2)]$ belong to the curves in figure 3, i.e. they correspond to fluid particles in the equator (border).

4. Conclusions

In this paper we have studied the stationary axisymmetric perfect fluid solution given by Wahlquist from several points of view. Firstly, we considered circularly orbiting test particles (co-rotating or counter-rotating with the fluid) in the equatorial plane of this solution and we were able to see that, as argued by Semerák, at limiting values of the particle's angular velocity, the 4-acceleration (or maintaining thrust) diverges ('counter-intuitively') to an *outward* direction or ('intuitively') to an *inward* direction, just being determined by their location relative to the photon orbit (which exists in the equatorial plane of the considered solution for certain ranges of values of the parameters). Specifically, the particle's acceleration behaves 'intuitively' below the photon orbit, whereas above the photon orbit it behaves 'intuitively' for counter-rotating particles and 'counter-intuitively' for co-rotating ones.

We have applied two major formalisms for the definition of the "general relativistic equivalents of inertial forces" to circularly orbiting particles in the equatorial plane of the Wahlquist solution —as far as we know, never before applied to a stationary non-static axisymmetric interior exact solution—, have analysed the results (particularly, for fluid particles) and have obtained a qualitatively similar behaviour in both acceleration (force) decompositions; in particular, a reversal in the sense of the centrifugal force, starting at the equator (border) at a certain rotation rate (different in each definition), and then, as the rotation rate increases from this critical value, rising a region which affects particles closer and closer to the centre of the body.

Both splittings of the inertia into different kinds of "inertial forces" in a general relativistic context illustrate the counter-intuitive features of dynamics of circular motion occurring in strong fields like the represented by the Wahlquist exact solution for a perfect fluid, in which it has been shown that the total acceleration (or maintaining thrust) of equatorial fluid particles (which is *outward* directed), when projected onto the equatorial geodesic distance unit vector, is, in absolute value, a decreasing function of the rotational parameter in a general rotation regime ($\kappa^2 > 1$), which is 'intuivive' from a Newtonian point of view; however, for slow rotation for normal objects ($\kappa^2 < 1$) it is an *increasing* function, but the geodesic *prolateness* also increases as the rotation rate increases, so that a larger (*inward*) total force corresponds to a smaller geodesic distance from the centre.

With this result, one is tempted to conclude that the boundary properties (total force at the equator, convexity, etc.) are related to the interior ones (*ellipticity*, etc.), as in the Newtonian case. However, as shown in a previous paper,[29] in general relativity one cannot directly extrapolate geometrical and dynamical boundary features to the interior of the fluid configuration.

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Rotational effects in the Kramer solution

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Abstract

Several dynamical properties of the Kramer solution for a rotating object (stationary axisymmetric) of perfect fluid are studied. We also present a definition of "rotational force" on fluid particles at the equator, following an analogy with the Newtonian classical description.

Keywords: exact solutions, self-gravitating systems, inertial forces

PACS: 04.20.Jb, 04.40.-b

Two major formalisms have been developed aiming at the interpretation of relativistic motion in terms of "Newtonian forces", and have been extended to stationary axisymmetric spacetimes.[1,2] In these terms, the motion of test particles in Kerr and Kerr-Newman spacetimes and approximation schemes valid for slow rotation have been analysed. However, to the best of our knowledge, those formalisms have not previously been applied to interior exact solutions.

Among these solutions (stationary axisymmetric perfect fluid, with a finite boundary of vanishing pressure, satisfying positivity energy conditions, and possessing no more Killing vectors than ∂_t and ∂_ϕ) there is one obtained by Kramer,[3]

$$\begin{aligned}2m\, ds^2 = &[\eta - 1 - b\cos\xi e^{-\eta}]dt^2 + [4(\eta-1) - 4b\cos\xi(e^{-\eta} - e^{-1})]dtd\phi \\ &+ [4(\eta-1) - 4b\cos\xi(e^{-\eta} + e^{\eta-2} - 2e^{-1})]d\phi^2 + \frac{d\eta^2}{\eta-1} + \frac{e^\eta}{b\cos\xi}d\xi^2\ ,\end{aligned}$$

where the boundary of the fluid has a simple expression in terms of elementary functions, $b\cos\xi = (1+\eta)e^\eta$, and the modulus of the vorticity vector at the *centre*, $(\eta = 1, \xi = 0)$, is an increasing function of b, $2e < b < 8.6$. Some geometrical properties of this solution have been analysed in Ref. 4.

We have studied the total acceleration of a test particle in circular motion in the Kramer spacetime, in particular, and following a suggestion given by Semerák,[5] we analyse, for circular orbiting test particles in the equatorial plane ($\xi = 0$), the dependence of the particle's acceleration on its angular velocity ω at limiting values, and its relation to the circular photon orbit appearing in this solution for $6.28 \leq b \leq 8.6$; and we have obtained that the 4-acceleration $\mathbf{a} = a^{\eta}(\omega)\,\partial_{\eta}$ diverges ('intuitively') to $-\infty$ (*inward*) below the photon orbit (when there is), whereas above the photon orbit, it diverges ('intuitively') to $-\infty$ for co-rotating particles, and diverges ('counter-intuitively') to $+\infty$ for counter-rotating particles.

When the two formalisms of "inertial forces" in general relativity, mentioned above, are applied to particles in the equatorial plane of the solution under consideration, one obtains a qualitatively similar behaviour in both acceleration (force) decompositions; in particular, the "centrifugal parts of acceleration" reverse sense at the equator at a certain rotation rate ($b \approx 6.55$ in the Abramowicz case, $b \approx 7.18$ in the Semerák case) and, as the rotation rate increases from these critical values, the reversal comprises a region of particles closer and closer to the centre of the body.

Finally, we present a definition of "rotational force" on fluid particles at the equator of the considered solution, following an analogy with the Newtonian classical description. Based on the Euler equation, $(\varepsilon + p)\mathbf{a} = -dp$, we introduce

$$\text{“weight” of polar column} \;\equiv\; \int_{s_{centre}}^{s_{pole}} (\varepsilon + p)\,|\mathbf{a}|\,ds\Big|_{\eta=1} = (p_{pole} - p_{centre})\big|_{\eta=1}$$

and, similarly,

$$\text{“weight” of equatorial column} \;=\; (p_{equator} - p_{centre})\big|_{\xi=0}\,.$$

Notice that $p_{pole} = p_{equator} = 0$, thus showing that

$$\text{weight of polar column} \;=\; \text{weight of equatorial column} \;=\; \frac{b}{e} - 2\,. \tag{1}$$

Let us now consider the "rotational ($\mathbf{r}$) - gravitational ($\mathbf{g}$) decomposition" of the acceleration $\mathbf{a}$ (force $\mathbf{f}$) of two fluid particles, one located at the pole ($\mathbf{p}$ subindex) and the other at the equator ($\mathbf{eq}$ subindex): suppose $\mathbf{a_p} = \mathbf{g_p}$ $(\mathbf{r_p} = \mathbf{0})$ and $\mathbf{f_{eq}} = -\mathbf{a_{eq}} = \mathbf{g_{eq}} + \mathbf{r_{eq}}$, with $|\mathbf{f_{eq}}| = |\mathbf{a_{eq}}| = q(|\mathbf{g_{eq}}| - |\mathbf{r_{eq}}|)$, $q = \pm 1$ ($\mathbf{g_{eq}}$ and $\mathbf{r_{eq}}$ anti-parallel) such that $\mathbf{f_{eq}}$ is outward directed. As $|\mathbf{a_p}| = \frac{\sqrt{2}}{4e}\sqrt{b^2 - 4e^2}$, we write b in terms of $|\mathbf{a_p}|$ (or $|\mathbf{g_p}|$), and Eq. (1) yields

$$\text{“weight” of polar column} = \frac{1}{e}\sqrt{2|\mathbf{g_p}|^2 + 4e^2} - 2\,.$$

We define, in analogy with Newtonian settings, $\mathbf{g_{eq}}$ and $\mathbf{r_{eq}}$ such that

$$\text{“weight” of equatorial column} = \left(\frac{1}{e}\sqrt{2|\mathbf{g_{eq}}|^2 + 4e^2} - 2\right)|1 - m|, \qquad m \equiv \frac{|\mathbf{r_{eq}}|}{|\mathbf{g_{eq}}|}\,,$$

from where, by Eq. (1),

$$\frac{1}{e}\sqrt{2|\mathbf{g_p}|^2 + 4e^2} - 2 = \left(\frac{1}{e}\sqrt{2|\mathbf{g_{eq}}|^2 + 4e^2} - 2\right)|1 - m|\,.$$

This yields

$$|\mathbf{g_{eq}}| = \frac{-4|\mathbf{a_{eq}}|k}{k^2 - 2/e^2\,|\mathbf{a_{eq}}|^2}, \qquad k \equiv \frac{1}{e}\sqrt{2|\mathbf{a_p}|^2 + 4e^2} - 2\,,$$

and for $|\mathbf{r_{eq}}|$,

case (i): $\quad q = 1, \quad |\mathbf{r_{eq}}| = |\mathbf{g_{eq}}| - |\mathbf{a_{eq}}|\,,$

case (ii): $\quad q = -1, \quad |\mathbf{r_{eq}}| = |\mathbf{g_{eq}}| + |\mathbf{a_{eq}}|\,.$

One can numerically see that in both cases $|\mathbf{g_{eq}}|$ and $|\mathbf{r_{eq}}|$ are increasing functions of the rotation rate (parameter b) [and of the "ellipticity" $\equiv \frac{l_{eq} - l_p}{l_{eq}} > 0$, l_{eq} and l_p the equatorial and polar (resp.) geodesic distances from the centre to the boundary]. In case (i) the "dilution factor" $m < 1$ is an increasing function of b, and in case (ii) $m > 1$ is a decreasing function of b; it follows that the defined inward part of the force ("rotational" or "gravitational"), which is always smaller than the outward part, increases more rapidly as the rotation rate (and the ellipticity) increases.

Accordingly, we find that, contrary to what happens in the Newtonian case, in general relativity the boundary properties (e.g., total force at the equator) are in general not related to the interior ones (e.g., ellipticity).

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Capítulo 4

Discusión integradora de las publicaciones presentadas

En la primera parte de este trabajo, Capítulo 2, se derivan propiedades generales de modelos relativistas estelares con rotación diferencial, en concreto se obtienen cotas en la velocidad de arrastre (que implica, en particular, la positividad de la densidad de momento angular) y cotas en la masa-energía total de rotación.

Las cotas en esta energía total de rotación son absolutamente generales y válidas para cualquier ley de rotación. Las cotas en la velocidad de arrastre dependen de la subyacente ley de rotación (la cual determina el modelo de rotación diferencial y ha de ser compatible con las ecuaciones de campo). Pero los resultados dados —en **Publicación I**— son para una clase muy general y físicamente importante de tales leyes de rotación (que incluye casi todos los perfiles de rotación de interés astrofísico, así como rotación rígida).

Sin embargo, en el caso de rotación lenta —estudiado en **Publicación II**—, resulta que la distribución de velocidad angular del fluido (o perfil de rotación) no es restringida por las ecuaciones de campo, a través de una posible ley de rotación. Y en el mismo artículo se refinan las cotas en la velocidad de arrastre. Por otra parte, la nota-Ref. 11 en **Publicación II** muestra cómo la ecuación componente $(t\,\phi)$ utilizada (en el caso de rotación lenta) corresponde a la ecuación que satisface la métrica general (cf. **Publicación I**) a primer orden en la rotación.

En un resultado —no previamente publicado— incluido aquí tras **Publicación II**, se analiza de forma exhaustiva esta ecuación, para obtener el comportamiento cualitativo de su solución, distribución de velocidad de arrastre, en una configuración con rotación lenta y diferencial. El argumento utilizado para probar estos resultados, y el propio comportamiento típico de estas funciones, daban las ideas de partida de resultados más generales, en **Publicaciones I** y **II**.

En cuanto a las cotas en la energía total de rotación, la prueba que damos en **Publicación II** en el caso límite de rotación lenta (prueba alternativa a la dada por Hartle) resulta ser generalizable en un régimen de rotación general (esto es, fuera de tal límite), Sec. V de **Publicación I**.

Este efecto arrastre de campos inerciales (objeto de estudio en **Publicaciones I** y **II**) es puramente relativista, y para poder comparar efectos relativistas de la rotación con resultados en el ámbito newtoniano, necesitamos estudiar soluciones particulares.

En **Publicación III** el análisis geométrico en relación con la forma y la convexidad de la superficie *borde* de la configuración de fluido de Kramer, muestra características (a velocidades altas de rotación del fluido) que se podrían interpretar como no-newtonianas; sin embargo, el comportamiento de las geodésicas en el *interior* de la configuración muestra el análogo de resultados newtonianos sobre convexidad.

Por otra parte, en **Publicaciones IV** y **V** hemos recogido ciertas propiedades cinemáticas y dinámicas de la solución —o familia de soluciones— de Wahlquist (incluida la solución de Kramer) interior para fluido perfecto y con rotación rígida, y las hemos relacionado con la geometría de estos espacio-tiempos interiores. En particular, se compara la fuerza total sobre partículas del fluido en el ecuador de la configuración —una propiedad *de frontera*— con la elipticidad de estas configuraciones, medida con distancias geodésicas desde el centro —una propiedad *interior*—, variando la velocidad de rotación del fluido.

Los resultados obtenidos en **Publicaciones IV** y **V** corroboran y completan **Publicación III** en la conclusión de que, a diferencia del caso newtoniano, en relatividad general, las propiedades geométricas y dinámicas de la frontera del fluido no pueden extrapolarse directamente al interior de la configuración de fluido.

En **Publicación IV** obtenemos asimismo efectos no-newtonianos en el movimiento de partículas del fluido en el plano ecuatorial, similares a los obtenidos en el caso estático, pero ahora la rotación de la fuente (en particular, el arrastre de los campos inerciales) hace que el efecto anómalo o no-newtoniano ocurra a partir de una cierta órbita límite que se desplaza con respecto a la correspondiente órbita en el caso estático con forme varía el parámetro de rotación del fluido; en concreto, este radio límite se hace menor a medida que la velocidad de rotación de la fuente aumenta.

Nota: La velocidad de un observador que localmente no rota (vista desde el infinito espacial), o lo que es lo mismo, de un observador con momento angular nulo (ZAMO) arrastrado por el campo gravitatorio del fluido —que coincide con la velocidad de arrastre del fluido— es denotada en **Publicación IV** por ω_d, en lugar de A, como en **Publicaciones I** y **II**.

Capítulo 5

Conclusiones

1) En un modelo estelar relativista con rotación diferencial, el arrastre de los campos inerciales ocurre en la misma dirección que la rotación del fluido estelar, cuando esta dirección es la misma para todas la partículas del fluido.

2) Para una gran clase de leyes de rotación, la distribución de velocidad angular del fluido estelar tiene signo (todas las partículas giran en el mismo sentido) y además ambos, la velocidad de arrastre y la densidad de momento angular, tienen este mismo signo.

3) En el caso particular de rotación rígida, la densidad de momento angular tiene el mismo signo que la (constante) velocidad angular del fluido.

4) El valor medio —con respecto a una densidad intrínseca— de la velocidad de arrastre es menor que el valor medio de la velocidad angular del fluido [independientemente de la ley de rotación, completamente en general].

5) La positividad y la cota superior de la energía total de rotación dadas por Hartle en el límite de rotación lenta (y diferencial) se generalizan fuera de este límite. En particular, la energía de rotación —que se demuestra positiva, y que decrece con el efecto arrastre (sobre lo que sería si este efecto fuese eliminado)— está acotada superiormente por el valor medio de la velocidad angular del fluido, y así, como era de esperar, crece con un incremento de la velocidad de rotación del fluido (en términos absolutos, sin signo).

6) Diferentes procedimientos ilustran las propiedades anti-intuitivas (desde un punto de vista newtoniano) de la dinámica del movimiento circular, que ocurren con campos gravitatorios fuertes (en rotación) como los representados por las soluciones exactas de Wahlquist y de Kramer para un fluido perfecto.

7) Contrariamente a lo que ocurre en el caso newtoniano, en el que las propiedades de frontera (fuerza total en el ecuador, convexidad, etc.) están relacionadas con las interiores, en relatividad general no podemos extrapolar directamente propiedades geométricas y dinámicas de la frontera al interior de la configuración de fluido.

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